We propose in this paper an analytical study of the temperature distribution in a solid subjected to moving heat sources. The power dissipated by the heat sources is considered non-uniform. The study was made in steady-state. The model is 3-D. It is valid regardless of the relative velocity of the source. We have considered three cases of semi-elliptic distribution of the power with: (1) the maximum at the center of the source, (2) the maximum at the inlet of the source, (3) the maximum at the output of the source. These configurations simulate the conformity imperfection of contact due to wear and/or the non-uniformity of contact pressure in frictional devices. We compare the temperature change for these different scenarios and for different relative velocities, considering the same total power dissipation. The reference case is that of a uniform source dissipating the same power.

Key words: analytical solution, advection-diffusion problem, non-uniform moving heat sources, sliding contact

Introduction

Mechanical devices such as brake discs, ball or roller bearings, gears, and many others, are subjected to friction, which generates a heat flux that can reach a few MW/m². This phenomenon leads to high temperature rises which causes damage of materials.

The calculation of temperature in solids subjected to frictional heating has been of great scientific interest over the past few decades. Since the pioneering works [1-5] concerning semi-infinite solids which are adiabatic outside the region heated by a moving heat source, many other research studies have been conducted. They deal with cooled semi-infinite bodies [6] and with rotating cylinder subjected to a heat source and surface cooling [6-13]. In certain industrial applications, the solids are provided with surface coating. Some studies have been carried out to analyse the effect of surface coating on the thermal behaviour of a solid subjected to friction process [14]. These research studies showed the evolution of flash temperatures (notion introduced in [15]) according to the main characteristic parameters, i.e., Peclet and Biot numbers. Peclet number is defined as: Pe = Vτ/α, where V is the relative velocity, τ – the characteristic length, and α – the thermal diffusivity of the material. Biot number is defined as: Bi = hL/λ, where h is the convective heat transfer coefficient, L – the characteristic length, and λ – the thermal conductivity of the material.
The theory of moving sources has long been used by several authors to simulate conditions of friction. The analytical solutions are often difficult to get because of: (1) the non-homogeneous boundary conditions (localized heating and surface cooling), (2) the relative motion (especially for the low values of Pe), and (3) the small size of the contact region in relation to other dimensions of solid.

The dimensionless parameter (the Peclet number) is generally introduced to characterize the thermal behaviour of solids subjected to moving heat sources. When this parameter is lower than a certain threshold, the thermal behaviour is comparable to that of a fixed source, which is largely studied in the literature. When this number is higher than another given threshold, the terms of heat diffusion in the source plane become negligible what allows determining analytical solutions. The great difficulty of the analytical computation of temperatures relates to intermediate values of the Pe ranging between these two thresholds. This problem was solved in [16] for a single source and in [17, 18] for multiple sources. Some other configurations involving heat sources have been studied in the literature [19, 20].

From available studies, it is clearly shown that the increase of the Pe involves a decrease in temperature elevation over the contact region for a given heat flux.

One of the difficulties encountered in the application of moving sources is the unknowledge of the spatial distribution of the power dissipated by these sources. Distributions of temperature and thermal gradients depend precisely of the spatial distribution of heat sources.

Often, the models proposed in the literature consider that the power is dissipated uniformly over the entire area of contact. In practice, the power dissipation can be extremely nonuniform due to: (1) the non-conformity of the contacts, (2) the wear, (3) defect of the mechanical guiding, etc. This phenomenon generates localized spots of heat, which can damage the material.

We propose in this paper an analytical study of the temperature distribution in a solid subjected to a non-uniform moving heat source. The study is made in steady state. The model is 3-D and it is valid regardless of the relative velocity of the source and its size.

**Mathematical model**

**Governing equations**

Let us consider an homogenous finite medium with thickness $e$ and area $2A \times 2B$ as shown in fig. 1. The plans $y = \pm B$ are assumed adiabatic. We consider a periodicity condition at the plans $x = -A$ and $x = A$. The surface $z = 0$ with $|x| \leq a$ and $|y| \leq b$ is subjected to a heat source with a flux density $q(x, y)$. The remainder of this area is assumed adiabatic. The temperature at the abscissa $z = e$ is zero, $T(x, y, z = e) = 0$. The solid is in relative motion with respect to the heat source at a constant velocity $V$ in the $x$-direction. We note $\lambda$ the thermal conductivity and $\alpha$ the thermal diffusivity of the solid.

In the referential related to the heat source, the steady-state temperature of the medium, $\bar{T}(x, y, z)$, is governed by the following equations:

$$\frac{\partial^2 \bar{T}}{\partial x^2} + \frac{\partial^2 \bar{T}}{\partial y^2} + \frac{\partial^2 \bar{T}}{\partial z^2} - \frac{V}{\alpha} \frac{\partial \bar{T}}{\partial x} = 0$$  

(1)

$$\left( T \right)_{(-A, y, z)} = \left( T \right)_{(A, y, z)} = \left( \frac{\partial T}{\partial x} \right)_{(-A, y, z)} = \left( \frac{\partial T}{\partial x} \right)_{(A, y, z)}$$  

(2)

$$\left( \frac{\partial T}{\partial y} \right)_{(x, -B, z)} = 0, \quad \left( \frac{\partial T}{\partial y} \right)_{(x, B, z)} = 0$$  

(3)
In eq. (1), the term \( \frac{V}{a} \left( \frac{\partial T}{\partial x} \right) \) presents the advective component of heat transfer in the medium. When this term is zero, only the conduction phenomenon takes place. The mathematical difficulty of this problem occurs when this term is non-zero. We develop an analytical solution of this problem which is valid regardless of the value of the term \( \frac{V}{a} \left( \frac{\partial T}{\partial x} \right) \).

General analytical solution

Taking into account the periodicity conditions with respect to x-direction, which is given by eq. (2), it is appropriate to apply the finite complex integral Fourier transform to eqs. (1)-(4) as:

\[
T^{**} = \frac{1}{2A} \int_{-A}^{A} T \exp \left( - \frac{j \pi n x}{A} \right) dx
\]  

where \( j \) is the imaginary number \( (j^2 = -1) \).

Considering the adiabatic conditions at the edges \( y = \pm B \) with respect to y-direction, it is appropriate to use the finite cosine integral Fourier transform as:

\[
T^{**} = \frac{1}{2B} \int_{-B}^{B} T^* \cos \left( \frac{\pi n y}{B} \right) dy
\]  

The application of the integral transforms (5) and (6) to equations (1-4) leads to a simple second order differential equation as:

\[
\frac{d^2 T^{**}}{dz^2} - \sigma_{mn}^2 T^{**} = 0
\]  

with

\[
\sigma_{mn} = \sqrt{\left( \frac{m\pi}{A} \right)^2 + \left( \frac{n\pi}{B} \right)^2 + \frac{j m \pi V}{\alpha A}}
\]

\[
-\lambda \left( \frac{dT^{**}}{dz} \right)_{(z=0)} = q^{**}, \quad T^{**} (z = c) = 0
\]  

The term \( \sigma_{mn} \) can be written under the following form: \( \sigma_{mn} = \rho_{mn} \rho_{mn} \) with:

\[
\rho_{mn} = \sqrt{\left( \frac{m\pi}{A} \right)^2 + \left( \frac{n\pi}{B} \right)^2 + \left( \frac{m\pi V}{\alpha A} \right)^2}
\]  

and

\[
\varphi_{mn} = \tan^{-1} \left[ \frac{m\pi V}{\alpha A} \left( \frac{1}{\left( \frac{m\pi}{A} \right)^2} + \left( \frac{n\pi}{B} \right)^2 \right) \right]
\]
The solution of eq. (7) requires distinguishing between the two following cases:

- For \(m = 0\) and \(n = 0\), for which \(\sigma_{00} = 0\). Then:
  \[ T_{00}^{**} = A_{00}z + B_{00} \]  
  \( \text{(9)} \)

Using the boundary conditions (8), we deduce:
  \[ T_{00}^{**} = -\frac{q_{00}^{**}}{\lambda} (z - c) \]
  \[ q_{00}^{**} = \frac{1}{4AB} \int_{-b}^{b} q(x, y)dx dy \]  
  \( \text{(9a)} \)

Equation (9a) is the 1-D solution, where \(q_{00}^{**}\) is the average heat flux density over the solid area \((2A \times 2B)\).

- For \(m \neq 0\) and/or \(n \neq 0\), for which \(\sigma_{mn} \neq 0\), then:
  \[ T_{mn}^{**} = A_{mn} \cosh(\sigma_{mn}z) + B_{mn} \sinh(\sigma_{mn}z) \]
  \[ q_{mn}^{**} = \frac{1}{4AB} \int_{-b}^{b} \int_{-a}^{a} q(x, y)e^{-jm\pi x/A} \cos \frac{n\pi y}{B} dx dy \]  
  \( \text{(10a)} \)

Expression of \(q_{mn}^{**}\) depends on the heat flux distribution of the source, \(q(x, y)\). We give in that follows the exact expression of \(q_{mn}^{**}\) for the flux distributions considered in this study.

Finally, we can decompose the solution of eq. (7) into four terms such as:
  \[ T^{**} = T_{00}^{**} + T_{m0}^{**} + T_{0n}^{**} + T_{mn}^{**} \]
  \( \text{(11)} \)

When the expressions of the four terms of eq. (11) are determined, we deduce the real temperature, \(T(x, y, z)\), by using the inverse integral transforms such as:

with
  \[ T = \Re \left\{ \sum_{n=0}^{\infty} \left( \sum_{m=0}^{\infty} T^{**} \cos \frac{m\pi x}{A} \right) \right\} \]  
  \( \text{(12)} \)

That gives:
  \[ T(x, y, z) = \frac{q_{00}^{**}(e - z)}{\lambda} + 2 \sum_{m=1}^{\infty} \Re \left\{ \frac{q_{m0}^{**} \sinh(\sigma_{m0}(e - z))e^{jm\pi x/A}}{\lambda \sigma_{m0} \cosh(\sigma_{m0}e)} \right\} + 2 \sum_{n=1}^{\infty} \Re \left\{ \frac{q_{0n}^{**} \sinh(\sigma_{0n}(e - z))e^{jm\pi x/A}}{\lambda \sigma_{0n} \cosh(\sigma_{0n}e)} \right\} \]  
  \( \text{(13)} \)

The notation \(\Re \{ \ldots \}\) denotes the real part.

**Complete solutions for different flux distributions \(q(x, y)\)**

We consider three cases of non-uniform flux distributions \(q(x, y)\) as shown in fig. 2, knowing that the case of a uniform heat flux \(q(x, y) = q_0\) is denoted Case 0.
The application of integral transforms (5) and (6) to the boundary condition (4) leads to expressions of \( q^* \) for the four cases studied under the form:

Case 0
\[
q_{00}^* = \frac{q_0 ab}{4AB}, \quad q_{mn}^* = \frac{q_1 J_0\left(\frac{m\pi a}{A}\right) \sin\left(\frac{n\pi b}{B}\right)}{m\pi(n\pi)}
\]
(15.0a)
\[
q_{00}^* = \frac{q_0 ab}{4AB}, \quad q_{mn}^* = \frac{q_0}{2m(n\pi)}, \quad q_{0n}^* = \frac{q_0}{4n}
\]

Case 1
\[
q_{00}^* = \frac{q_1 \pi ab}{4AB}, \quad q_{mn}^* = \frac{q_1 J_0\left(\frac{m\pi a}{A}\right) \sin\left(\frac{n\pi b}{B}\right)}{2m(n\pi)}
\]
(15.1a)
\[
q_{00}^* = \frac{q_1 \pi ab}{4AB}, \quad q_{mn}^* = \frac{q_1}{4n}, \quad q_{0n}^* = \frac{q_1 J_0\left(\frac{m\pi a}{A}\right) \sin\left(\frac{n\pi b}{B}\right)}{4n}
\]

Case 2
\[
q_{00}^* = \frac{q_2 \pi ab}{4AB}, \quad q_{mn}^* = \frac{q_2 J_0\left(\frac{m\pi a}{A}\right) \sin\left(\frac{n\pi b}{B}\right) e^{jm\pi a/A}}{4m(n\pi)}
\]
(15.2a)
\[
q_{00}^* = \frac{q_2 \pi ab}{4AB}, \quad q_{mn}^* = q_2 \frac{J_0\left(\frac{m\pi a}{A}\right) \sin\left(\frac{n\pi b}{B}\right) e^{jm\pi a/A}}{4n}, \quad q_{0n}^* = q_2 \frac{a}{4n}
\]

Case 3
\[
q_{00}^* = \frac{q_3 \pi ab}{4AB}, \quad q_{mn}^* = q_3 J_0\left(\frac{m\pi a}{A}\right) e^{-jm\pi a/A} \frac{\sin\left(\frac{n\pi b}{B}\right)}{4m(n\pi)}
\]
(15.3a)
\[
q_{00}^* = \frac{q_3 \pi ab}{4AB}, \quad q_{mn}^* = q_3 J_0\left(\frac{m\pi a}{A}\right) e^{-jm\pi a/A} \frac{\sin\left(\frac{n\pi b}{B}\right)}{4n}, \quad q_{0n}^* = q_3 \frac{a}{4n}
\]

where \( J_1 \) is the Bessel function of the first kind of order 1 and \( H_1 \) is the Struve function of order 1.
Results and discussions

We study the effect of spatial distribution of the heat dissipated by the source on the temperature for different values of Peclet number: $\text{Pe} = \frac{VA}{\alpha} = 0, 20, \text{and} 200$. The total power is the same for each distribution, that requires: $q_1 = q_2 = q_3 = 4q_0/\pi$. The temperature is given in a dimensionless form: $T^+ = T(q_0A/\beta)$. We fixed arbitrarily: $B = A, b = a, e = 0.5A, a/A = 0.1$ (i.e. $ab/AB = 1/100$). The number of terms necessary to ensure the convergence of series is, in this case, equal to 150. It should be noted that this number decreases with the increase of the ratio $ab/AB$.

Figure 3 shows the change of the surface temperature of each studied case and for the three values of $\text{Pe}$. The heating zone is between $-0.1$ and $0.1$. We can note two tendencies common to all the cases considered here: (1) the maximum surface temperature moves out of the source as the value of the $\text{Pe}$ increases, and (2) the amplitude of the temperature peak (flash temperature) in the zone subjected to the heat source decreases with the increase of the value of the $\text{Pe}$.

![Figure 3. Dimensionless surface temperature $T^+(x, 0, 0)$ for the four studied cases and for different values of $\text{Pe} = 0, 20, \text{and} 200$](image-url)
Maximum surface temperatures are summarized in tab. 1. The highest surface temperature at $Pe = 0$ is obtained for the semi-elliptic centered source (case 1). We can note that for $Pe = 0$ the cases of left and right eccentric sources give logically the same temperature. When the value of $Pe$ increases, the eccentric source at the right (case 3) gives the highest surface temperature. This result means that in the case of practical applications, such as brake pads, for example, temperatures will be much higher than the pads carry more at the output contact region. Here, the temperature difference is not significant because the semi-elliptical profile is relatively flat, but in the case of sharp profiles, this difference becomes more marked.

Figure 4 shows the temperature change in the depth direction of the material at the abscissa $x = 0$ and $y = 0$ (i.e., at the center of solid) in the case 1. Temperatures are plotted for three values of the $Pe$. We can note that for $Pe$ close to zero, the 3-D effects are still present, but as soon as $Pe$ becomes high, 3-D effects are localized in a thin region, which is located in the vicinity of the heated face ($z = 0$). Beyond this region, the temperature becomes 1-D ($T = T(z)$ only). This occurs at $z/A \geq 0.2$ for $Pe = 20$ and at $z/A \geq 0.06$ for $Pe = 200$.

### Conclusions

Analytical solutions are presented in this paper to study the influence of non-uniformity of the distribution of heat sources on the thermal behavior of solids in relative motion (friction, welding, laser ...). These solutions are explicit and easy to use, since the functions which they use are available in different software such as Maple, Mathematica, etc.

The proposed analytical method can be used for other configurations of the distribution of heat sources that those considered in this study.

### Table 1. Comparison of maximum surface temperatures

<table>
<thead>
<tr>
<th>$Pe$</th>
<th>Case 0</th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.10345</td>
<td>0.11474</td>
<td>0.11129</td>
<td>0.11129</td>
</tr>
<tr>
<td>20</td>
<td>0.08331</td>
<td>0.09061</td>
<td>0.08522</td>
<td>0.09188</td>
</tr>
<tr>
<td>200</td>
<td>0.04522</td>
<td>0.04563</td>
<td>0.04308</td>
<td>0.04903</td>
</tr>
</tbody>
</table>

### Figure 4. Dimensionless temperature $T'(0, 0, z)$ in the depth for the case 1 and for different values of $Pe = 0, 20, 200$

### Nomenclature

- $A$ – semi-width of the solid, [m]
- $a$ – semi-width of the source, [m]
- $B$ – semi-length of the solid, [m]
- $b$ – semi-length of the source, [m]
- $e$ – thickness of the solid, [m]
- $H$ – Struve function
- $J$ – Bessel function
- $Pe$ – Peclet number, $(= VA/\lambda)$
- $q$ – heat flux density, [Wm$^{-2}$]
- $T$ – temperature, [K]
- $T'$ – dimensionless temperature $(= T(q_0 A \lambda))$
- $V'$ – velocity, [ms$^{-1}$]
- $x, y, z$ – Cartesian co-ordinates, [m]
- $\alpha$ – thermal diffusivity, [m$^2$s$^{-1}$]
- $\lambda$ – thermal conductivity, [Wm$^{-1}$K$^{-1}$]
References