Approximation Theorems for $q$-Bernstein-Kantorovich Operators

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Abstract. In the present paper we introduce a $q$-analogue of the Bernstein-Kantorovich operators and investigate their approximation properties. We study local and global approximation properties and Voronovskaja type theorem for the $q$-Bernstein-Kantorovich operators in case $0 < q < 1$.

1. Introduction

In the last two decades interesting generalizations of Bernstein polynomials were proposed by Lupas [15] and by Phillips [20]. Generalizations of the Bernstein polynomials based on the $q$-integers attracted a lot of interest and was studied widely by a number of authors. A survey of the obtained results and references on the subject can be found in [19]. Recently some new generalizations of well known positive linear operators, based on $q$-integers were introduced and studied by several authors, see [23], [5], [6], [8], [21], [22], [16].

The classical Kantorovich operator $B^*_n, n = 1, 2, ...$ is defined by (cf. [14])

$$B^*_n(f; x) := (n + 1) \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) x^k (1 - x)^{n-k} \int_{k/n+1}^{(k+1)/n+1} f(t) \, dt$$

$$= \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) x^k (1 - x)^{n-k} \int_{0}^{1} f\left( \frac{k + t}{n+1} \right) \, dt, \quad f : [0, 1] \to \mathbb{R}. \quad (1)$$

These operators have been extensively considered in the mathematical literature. Also, a number of generalizations have been introduced by different authors (see, for instance [24], [25], [26]).

In this paper, inspired by (1), we introduce a $q$-type generalization of Bernstein-Kantorovich polynomial operators as follows.

$$B^*_n(f; x) := \sum_{k=0}^{n} p_{n,k}(q; x) \int_{0}^{1} f\left[ k + q^k t \right] \left[ n + 1 \right] \, d_q t,$$

where $f \in C [0, 1], \, 0 < q < 1$. 

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2010 Mathematics Subject Classification. Primary 41A46 ; Secondary 33D99, 41A25

Keywords. $q$-integers, positive operator, $q$-Bernstein-Kantorovich operator

Received: 26 November 2011; Accepted: 8 December 2012

Communicated by Gradimir Milovanovic

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The paper is organized as follows. In Section 2, we give standard notations that will be used throughout the paper, introduce $q$-Bernstein-Kantorovich operators and evaluate the moments of $B_{n,q}^*$. In Section 3 we study local and global convergence properties of the $q$-Bernstein-Kantorovich operators and prove Voronovskaja-type asymptotic formula. In the final section we give statistical approximation result for the $q$-Bernstein-Kantorovich operators.

2. $q$-Bernstein-Kantorovich operators

Let $q > 0$. For any $n \in \mathbb{N} \cup \{0\}$, the $q$-integer $[n]_q$ is defined by

$$[n]_q := 1 + q + \ldots + q^{n-1}, \quad [0] := 0;$$

and the $q$-factorial $[n]_q! = [n]_q!$ by

$$[n]_q! := [1]_q[2]_q\ldots[n]_q, \quad [0]_q! := 1.$$

For integers $0 \leq k \leq n$, the $q$-binomial coefficient is defined by

$$\left[ \begin{array}{c} n \\ k \end{array} \right] := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$  

The $q$-analogue of integration in the interval $[0, A]$ (see [13]) is defined by

$$\int_0^A f(t)\, dq^t := A(1-q) \sum_{n=0}^{\infty} f(Aq^n)\, q^n, \quad 0 < q < 1.$$

Let $0 < q < 1$. Based on the $q$-integration we propose the Kantorovich type $q$-Bernstein polynomial as follows.

$$B_{n,q}^*(f,x) = \sum_{k=0}^n p_{n,k}(q;x) \int_0^1 f([k] + q^k t)\, dq^t, \quad 0 \leq x \leq 1, n \in \mathbb{N}$$

where

$$p_{n,k}(q;x) := \left[ \begin{array}{c} n \\ k \end{array} \right] x^k (1-x)^{n-k}_q, \quad (1-x)_q^n := \prod_{i=0}^{n-1} (1-q^i x).$$

It can be seen that for $q \to 1^-$ the $q$-Bernstein-Kantorovich operator becomes the classical Bernstein-Kantorovich operator.

Lemma 2.1. For all $n \in \mathbb{N}$, $x \in [0,1]$ and $0 < q \leq 1$ we have

$$B_{n,q}^*(t^n, x) = \sum_{j=0}^m \left( \begin{array}{c} m \\ j \end{array} \right) \frac{[n]_q!}{[n+1]_q!} \frac{[m-j]_q!}{[m-j+1]_q!} \sum_{i=0}^{m-j} \left( \begin{array}{c} m-j \\ i \end{array} \right) (q^n - 1)^i B_{n,q}^*(t^i, x).$$  

(2)
For all \( n \), taking into account (2), by direct computation, we obtain explicit formulas for \( B_{n,q}^r \).

**Proof.** The recurrence formula can be derived by direct computation.

\[
B_{n,q}^r(t^n, x) = \sum_{k=0}^{n} p_{n,k}(q; x) \sum_{j=0}^{m} \binom{m}{j} \frac{q^j (n-j) t^{m-j}}{[n+1]^m [m-j+1]} d_q t
\]

\[
= \sum_{k=0}^{n} p_{n,k}(q; x) \sum_{j=0}^{m} \binom{m}{j} \frac{q^j (m-j) [k]^j}{[n+1]^m [m-j+1]}
\]

\[
= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j}{[n+1]^m [m-j+1]} \sum_{k=0}^{n} (q^j - 1) [k]^j p_{n,k}(q; x)
\]

\[
= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j}{[n+1]^m [m-j+1]} \sum_{k=0}^{n} \sum_{i=0}^{m-j} \binom{m-j}{i} (q^j - 1)^i [k]^i p_{n,k}(q; x)
\]

\[
= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j}{[n+1]^m [m-j+1]} \sum_{k=0}^{n} \sum_{i=0}^{m-j} \binom{m-j}{i} (q^j - 1)^i [k]^i p_{n,k}(q; x)
\]

\[
= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^j}{[n+1]^m [m-j+1]} \sum_{k=0}^{n} \sum_{i=0}^{m-j} \binom{m-j}{i} (q^j - 1)^i B_{n,q}^r(t^i, x).
\]

**Lemma 2.2.** For all \( n \in \mathbb{N} \), \( x \in [0,1] \) and \( 0 < q \leq 1 \) we have

\[
B_{n,q}^r(1, x) = 1, \quad B_{n,q}^r(t, x) = \frac{2q [n]}{[2][n+1]} x + \frac{1}{[2][n+1]} \]

\[
B_{n,q}^r(t^2, x) = \frac{q(q+2)}{[3]} \frac{2q [n]}{[2][n+1]} x^2 + \frac{4q^2 + q^3}{[2][3]} \frac{[n]}{[n+1]^2} x^2 + \frac{1}{[3][n+1]^2}.
\]

**Proof.** Taking into account (2), by direct computation, we obtain explicit formulas for \( B_{n,q}^r(t, x) \) and \( B_{n,q}^r(t^2, x) \) as follows.

\[
B_{n,q}^r(t, x) = \frac{1}{[n+1][2]} (B_{n,q}^r(1, x) + (q^n - 1) B_{n,q}^r(t, x)) + \frac{[n]}{[n+1]} B_{n,q}^r(t, x)
\]

\[
= \left( \frac{q^n - 1}{[2][n+1]} + \frac{[n]}{[n+1]} \right) x + \frac{1}{[2][n+1]} = \frac{2q}{[2][n+1]} x + \frac{1}{[2][n+1]}
\]

and

\[
B_{n,q}^r(t^2, x) = \frac{1}{[3][n+1]^2} \left( B_{n,q}^r(1, x) + 2 (q^n - 1) B_{n,q}^r(t, x) + (q^n - 1)^2 B_{n,q}^r(t^2, x) \right)
\]

\[
+ \frac{2 [n]}{[2][n+1]^2} \left( B_{n,q}^r(t, x) + (q^n - 1) B_{n,q}^r(t^2, x) \right) + \frac{[n]^2}{[n+1]^2} B_{n,q}^r(t^2, x)
\]

\[
= \frac{1}{[3][n+1]^2} + \frac{[n]^2}{[n+1]^2} + \frac{2 [n]}{[2][n+1]^2} + \frac{(q^n - 1)^2}{[3][n+1]^2} \left( 1 - \frac{1}{[n]} \right) x^2
\]

\[
+ \left( \frac{[n]^2}{[n][n+1]^2} + \frac{2 [n]}{[2][n+1]^2} + \frac{(q^n - 1)^2}{[3][n][n+1]^2} + \frac{2 [n]}{[2][n+1]^2} + \frac{(q^n - 1)^2}{[3][n+1]^2} \right) x
\]

\[
= \frac{2q + 3q^2 + q^3}{[2][3]} \frac{q [n]}{[n+1]^2} x^2 + \frac{4q + 7q^2 + q^3}{[2][3]} \frac{[n]}{[n+1]^2} x^2 + \frac{1}{[3][n+1]^2}.
\]
Remark 2.3. It is observed from the above lemma that for \( q = 1 \), we get the moments of the Bernstein-Kantorovich operators.

Lemma 2.4. For all \( n \in \mathbb{N}, x \in [0, 1] \) and \( 0 < q \leq 1 \) we have

\[
B_{n,q}^*\left((t-x)^2, x\right) \leq \frac{4}{[n]} \left(x(1-x) + \frac{1}{[n]}\right), \quad B_{n,q}^*\left((t-x)^4, x\right) \leq \frac{C}{[n]^2} \left(x(1-x) + \frac{1}{[n]^2}\right),
\]

where \( C \) is a positive absolute constant.

Proof. Note that estimation of the moments for the \( q \)-Bernstein operators is given in [17]. The proof is based on the estimations of the second and fourth order central moments of the \( q \)-Bersntein polynomials.

\[
B_{n,q}^*\left((t-x)^2, x\right) = \frac{1}{[n]} x(1-x), \quad B_{n,q}^*\left((t-x)^4, x\right) \leq \frac{C}{[n]^2} x(1-x).
\]

Indeed

\[
B_{n,q}^*\left((t-x)^2, x\right)
= \sum_{k=0}^{n} p_{n,k}(q;x) \left( \frac{[k] + q^t [n]}{[n+1]} - x \right)^2 d_q t
= \sum_{k=0}^{n} p_{n,k}(q;x) \left( \frac{[k] + q^t [n]}{[n+1]} - x \right)^2 d_q t
\leq 2 \sum_{k=0}^{n} p_{n,k}(q;x) \left( \frac{[k] + q^t [n]}{[n+1]} - x \right)^2 d_q t
\leq \frac{4}{[3][n+1]^2} + \frac{2}{[n]^2} x(1-x) \leq \frac{4}{[n]} \left(x(1-x) + \frac{1}{[n]}\right).
\]

A similar calculus reveals:

\[
B_{n,q}^*\left((t-x)^4, x\right)
= \sum_{k=0}^{n} p_{n,k}(q;x) \left( \frac{[k] + q^t [n]}{[n+1]} - x \right)^4 d_q t
\leq \frac{32}{[5][n+1]^4} + \frac{4}{[n]^2} C x (1-x) \leq \frac{C}{[n]^2} \left(x(1-x) + \frac{1}{[n]^2}\right).
\]

Lemma 2.5. Assume that \( 0 < q_n < 1, q_n \to 1 \) and \( q_n^m \to a \) as \( n \to \infty \). Then we have

\[
\lim_{n \to \infty} [n]_{q_n} B_{n,q_n}^*\left(t-x; x\right) = -\frac{1}{2} + a + \frac{1}{2},
\]

\[
\lim_{n \to \infty} [n]_{q_n} B_{n,q_n}^*\left((t-x)^2; x\right) = -\frac{1}{3} x^2 - \frac{2}{3} a x + x.
\]

Proof. To prove the lemma we use formulas for \( B_{n,q_n}^*\left(t; x\right) \) and \( B_{n,q_n}^*\left(t^2; x\right) \) given in Lemma 2.2.
3. Local and global approximation

We begin by considering the following K-functional:

\[ K_2 \left( f, \delta^2 \right) \triangleq \inf \left\{ \| f - g \| + \delta^2 \| g'' \| : g \in C^2 [0,1] \right\}, \quad \delta \geq 0, \]

where

\[ C^2 [0,1] \triangleq \{ g : g, g', g'' \in C [0,1] \}. \]

Then, in view of a known result [7], there exists an absolute constant \( C_0 > 0 \) such that

\[ K_2 \left( f, \delta^2 \right) \leq C_0 \omega_2 \left( f, \delta \right) \tag{3} \]

where

\[ \omega_2 \left( f, \delta \right) \triangleq \sup_{0 < h < \delta} \sup_{x \in [0,1]} \left| f(x + h) - 2f(x) + f(x - h) \right| \]

is the second modulus of smoothness of \( f \in C[0,1] \).

Our first main result is stated below.

**Theorem 3.1.** There exists an absolute constant \( C > 0 \) such that

\[ \left| B_{n,q}^* (f; x) - f(x) \right| \leq C \omega_2 \left( f, \frac{\delta_n(x)}{[n]} \right) + \omega_1 \left( f, \frac{(1 + q x^2) x - 1}{[2][n + 1]} \right), \]

where \( f \in C[0,1], \delta_n(x) = q^2(x) + \frac{1}{[n]}, 0 \leq x \leq 1 \) and \( 0 < q < 1 \).

**Proof.** Let

\[ \tilde{B}_{n,q}^* (f; x) = B_{n,q}^* (f; x) + f(x) - f(a_n x + b_n), \]

where \( f \in C[0,1], a_n = \frac{2a}{1 + q [n + 1]} \) and \( b_n = \frac{1}{1 + q [n + 1]} \). Using the Taylor formula

\[ g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-s) g''(s) \, ds, \quad g \in C^2[0,1], \]


we have
\[ \overline{B}_{n,q}^* (g;x) = g(x) + B_{n,q}^* \left( \int_x^\infty (t-s) g''(s) \, ds; x \right) - \int_x^{a_nx+b_n} (a_n x + b_n - s) g''(s) \, ds, \quad g \in C^2 [0,1]. \]

Hence
\[ \left| \overline{B}_{n,q}^* (g;x) - g(x) \right| \leq B_{n,q}^* \left( \int_x^\infty |t-s| |g''(s)| \, ds; x \right) + \int_x^{a_nx+b_n} |a_n x + b_n - s| |g''(s)| \, ds \]
\[ \leq \|g''\| B_{n,q}^* (t-x)^2; x) + \|g''\| (a_n x + b_n - x)^2 \]
\[ \leq \|g''\| \left\{ \frac{4}{[n]} (x (1-x) + \frac{1}{[n]} \right\} + 4 \frac{[n]}{[n]^2} x^2 + \frac{2}{[n]^2} \}
\[ = \frac{10}{[n]} \delta_n(x) \|g''\|. \] (4)

Using (4) and the uniform boundedness of \( \overline{B}_{n,q}^* \) we get
\[ \left| B_{n,q}^* (f;x) - f(x) \right| \leq \left| \overline{B}_{n,q}^* (f-g;x) \right| + \left| B_{n,q}^* (g;x) - g(x) \right| + \left| f(x) - g(x) \right| + \left| f(a_n x + b_n) - f(x) \right|
\[ \leq 4 \|f-g\| + \frac{10}{[n]} \delta_n(x) \|g''\| + \omega (f, |(a_n-1) x + b_n|). \]

Taking the infimum on the right hand side over all \( g \in C^2 [0,1] \), we obtain
\[ \left| B_{n,q}^* (f;x) - f(x) \right| \leq 10K_2 \left( f, \frac{\delta_n(x)}{[n]} \right) + \omega (f, |(a_n-1) x + b_n|), \]

which together with (3) gives the proof of the theorem.

**Corollary 3.2.** Assume that \( q_n \in (0,1) \), \( q_n \to 1 \) as \( n \to \infty \). For any \( f \in C^2 [0,1] \) we have
\[ \lim_{n \to \infty} \left\| B_{n,q_n}^* (f) - f \right\| = 0. \]

We next present the direct global approximation theorem for the operators \( B_{n,q}^* \). In order to state the theorem we need the weighted \( K \)-functional of second order for \( f \in C[0,1] \) defined by
\[ K_{2,\omega} \left( f, \delta^2 \right) := \inf \left\{ \|f - g\| + \delta^2 \|\varphi^2 g''\| : g \in W^2(\varphi) \right\}, \quad \delta \geq 0, \quad \varphi^2(x) = x(1-x) \]
where
\[ W^2(\varphi) := \left\{ g \in C[0,1] : g' \in AC[0,1], \quad \varphi^2 g'' \in C[0,1] \right\}, \]
and \( g' \in AC[0,1] \) means that \( g \) is differentiable and \( g' \) is absolutely continuous in \([0,1]\). Moreover, the Ditzian-Totik modulus of second order is given by
\[ \omega_{2}^\varphi(f, \delta) := \sup_{0 < h \leq \delta} \sup_{x \in [0,1]} |f(x - \varphi(x) h) - 2f(x) + f(x + \varphi(x) h)|. \]

It is well known that the \( K \)-functional \( K_{2,\omega} \left( f, \delta^2 \right) \) and the Ditzian-Totik modulus \( \omega_{2}^\varphi(f, \delta) \) are equivalent (see [7]).

Now we state our next main result.
Theorem 3.3. There exists an absolute constant $C > 0$ such that
\[
\|B_{n,q}^\ast (f) - f\| \leq C\omega_2^q \left( f, \frac{1}{\sqrt{\|f\|}} \right) + \omega_2 \left( f, \frac{1}{\|f\|} \right),
\]
where $f \in C[0,1]$, $0 < q < 1$, $\varphi^2 (x) = x (1 - x)$, $\psi (x) = 2x + 1$.

Proof. Let
\[
B_{n,q}^\ast (f; x) = B_{n,q}^\ast (f; x) + f \left( ax + b_n \right),
\]
where $f \in C[0,1]$, $a_n = \frac{2q}{1 + q}$, $b_n = \frac{1}{1 + q}$. Using the Taylor formula
\[
g \left( t \right) = g \left( x \right) + g' \left( x \right) (t - x) + \int_x^t (t - s) g'' (s) \, ds, \quad g \in W^2 \left( \varphi \right),
\]
we have
\[
\bar{B}_{n,q}^\ast (g; x) = g \left( x \right) + B_{n,q}^\ast \left( \int_x^t (t - s) g'' (s) \, ds; x \right) - \int_x^t (a_n x + b_n - s) g'' (s) \, ds, \quad g \in W^2 \left( \varphi \right).
\]
Hence
\[
\|\bar{B}_{n,q}^\ast (g; x) - g \left( x \right)\| \leq B_{n,q}^\ast \left( \left\| \int_x^t |t - s| |g'' (s)| \, ds \right\|; x \right) + \left\| \int_x^t |a_n x + b_n - s| |g'' (s)| \, ds \right\|.
\]
Because the function $\delta_n^2$ is concave on $[0,1]$, we have for $u = t + r(x-t)$, $r \in [0,1]$, the estimate
\[
\frac{|t - s|}{\delta_n^2 (s)} = \frac{r |x - t|}{\delta_n^2 (s)} \leq \frac{r |x - t|}{\delta_n^2 (t) + r (\delta_n^2 (x) - \delta_n^2 (t))} \leq \frac{|x - t|}{\delta_n^2 (x)}.
\]
Hence, by (5), we find
\[
\|\bar{B}_{n,q}^\ast (g; x) - g \left( x \right)\| \leq \|\delta_n^2 g''\| B_{n,q}^\ast \left( \left\| \int_x^t |t - s| |g'' (s)| \, ds \right\|; x \right) + \|\delta_n^2 g''\| \left\| \int_x^t |a_n x + b_n - s| / \delta_n^2 (s) \, ds \right\|
\]
\[
\leq \|\delta_n^2 g''\| \left( B_{n,q}^\ast \left( (t - x)^2; x \right) + (a_n x + b_n - x)^2 \right)
\]
\[
\leq \|\delta_n^2 g''\| \left( 4 \left\| \frac{1}{n} \right\| \left( x (1 - x) + \frac{1}{n} \right) + 4 \left\| \frac{1}{n^2} \right\| x^2 + 2 \left\| \frac{2}{n^2} \right\| \right)
\]
\[
\leq \|\delta_n^2 g''\| \left( 10 \left\| \frac{1}{n} \right\| \left( x (1 - x) + \frac{1}{n} \right) \right) = \frac{10}{|n|} \|\delta_n^2 g''\|.
\]
Since
\[
\|\varphi^2 g''\| \leq \|\varphi^2 g''\| + \frac{1}{|n + 1|} \|g''\|
\]
we have
\[
\|\bar{B}_{n,q}^\ast (g; x) - g \left( x \right)\| \leq \frac{10}{|n|} \left( \|\varphi^2 g''\| + \frac{1}{|n|} \|g''\| \right).
\]
Using (6) and the uniform boundedness of $B_{n,q}$ we get
\[ |B_{n,q}(f;x) - f(x)| \leq |\tilde{B}_{n,q}(f-g;x) + \tilde{B}_{n,q}(g-x) + f(x) - g(x)| + |f(a_n x + b_n) - f(x)| \]
\[ \leq 4\|f-g\| + \frac{10}{\|n\|} \left( \|q^2 g''\| + \frac{1}{\|n\|} \|g''\| \right) + |f(a_n x + b_n) - f(x)|. \]

Taking the infimum on the right hand side over all $g \in W^2(\varphi)$, we obtain
\[ |B_{n,q}(f;x) - f(x)| \leq 10K_{2,\varphi} \left( f; \frac{1}{\|n\|} \right) + |f(a_n x + b_n) - f(x)|. \]

On the other hand
\[ |f(a_n x + b_n) - f(x)| = |f(x + \varphi(x)(a_n - 1) x + b_n) - f(x)| \]
\[ \leq \sup \left| \frac{1 + q^{r+1}}{\varphi(x)[n+1]} x + \frac{1}{2} [n+1] \varphi(x) \right| - f(x) \]
\[ \leq \omega_x^\varphi \left( f; \frac{B_{n,q}(t;x) - x}{\varphi(x)} \right) \leq \omega_x^\varphi \left( f; \frac{2x + 1}{2} \varphi(x) \right). \]

Hence, by (7) and (8), using the equivalence of $K_{2,\varphi} \left( f; \frac{1}{\|n\|} \right)$ and the Ditzian-Totik modulus $\omega_x^\varphi \left( f, \sqrt{\|n\|} \right)$ we get the desired estimate.

Next we prove Voronovskaja type result for $q$-Bernstein-Kantorovich operators.

**Theorem 3.4.** Assume that $q_n \in (0,1)$, $q_n \to 1$ and $q_n^2 \to a$ as $n \to \infty$. For any $f \in C^2 [0,1]$ the following equality holds
\[ \lim_{n \to \infty} [n]_{q_n} \left( B_{n,q_n}(f;x) - f(x) \right) = f'(x) \left( -\frac{1}{2} x^2 - \frac{1}{2} x \right) + f''(x) \left( \frac{1}{3} x^2 - \frac{2}{3} ax^2 + x \right) \]
uniformly on $[0,1]$.

**Proof.** Let $f \in C^2 [0,1]$ and $x \in [0,1]$ be fixed. By the Taylor formula we may write
\[ f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + r(t;x)(t-x)^2, \]
where $r(t;x)$ is the Peano form of the remainder, $r(;x) \in C[0,1]$ and $\lim_{t \to x} r(t;x) = 0$. Applying $B_{n,q_n}$ to (9) we obtain
\[ [n]_{q_n} \left( B_{n,q_n}(f;x) - f(x) \right) = f'(x) [n]_{q_n} B_{n,q_n}(t-x) + \frac{1}{2} f''(x) [n]_{q_n} B_{n,q_n}(t-x)^2 + [n]_{q_n} B_{n,q_n}(r(t;x)(t-x)^2). \]

By the Cauchy-Schwartz inequality, we have
\[ B_{n,q_n}(r(t;x)(t-x)^2) \leq \sqrt{B_{n,q_n}(r^2(t;x)^2)} \sqrt{B_{n,q_n}((t-x)^4)} \]

Observe that $r^2(x;x) = 0$ and $r^2(;x) \in C[0,1]$. Then it follows from Corollary 3.2 that
\[ \lim_{n \to \infty} B_{n,q_n}(r^2(t;x)^2) = r^2(x;x) = 0 \]
uniformly with respect to $x \in [0,1]$. Now from (10), (11) and Lemma 2.5 we get immediately
\[ \lim_{n \to \infty} [n]_{q_n} B_{n,q_n}(r(t;x)(t-x)^2) = 0. \]

The proof is completed.
4. Statistical approximation

At this moment, we recall the concept of statistical convergence. The density of a subset $K$ of $\mathbb{N}$ is given by $\delta(K) := \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{K}(k)$, whenever the limit exists, where $\chi_{K}$ is the characteristic function of $K$. A sequence $x = \{x_n\}_{n \in \mathbb{N}}$ is said to be statistically convergent to $L$ if for any $\varepsilon > 0$, $\delta\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0$ and it is denoted by $st \lim x = L$ (see[10]).

Assume that $\{q_n\}_{n \in \mathbb{N}}$ be sequence from $(0, 1]$ such that

$$st \lim_{n} q_n = 1. \tag{12}$$

Observe that for any sequence $\{q_n\}_{n \in \mathbb{N}} \subset (0, 1]$, satisfying (12) and for fixed $x \in [0, 1]$, we have

$$st \lim_{n} \frac{\delta_n(x)}{[n]_{q_n}} = st \lim_{n} \left[\frac{(1 + q_n^{n+1})x - 1}{[2]_{q_n}[n + 1]_{q_n}}\right] = 0, \tag{13}$$

which yields

$$st \lim_{n} \frac{f}{\sqrt{\delta_n(x)}} = 0, \tag{14}$$

and

$$st \lim_{n} \frac{f}{\sqrt{\delta_n(x)}} \left(\frac{(1 + q_n^{n+1})x - 1}{[2]_{q_n}[n + 1]_{q_n}}\right) = 0 \tag{15}$$

respectively. So, Theorem 3.1 gives the following statistical approximation theorem.

**Theorem 4.1.** Assume that, $\{q_n\}_{n \in \mathbb{N}}$ is a sequence satisfying (12). Then, for all $f \in C[0, 1]$ and fixed $x \in [0, 1]$, we have

$$st \lim_{n} B_{q_n}^{n}(f; x) - f(x) = 0.$$

**References**

[22] C. Radu, Statistical approximation properties of Kantorovich operators based on $q$-integers, Creative Math and Inf. 17 (2008), 75-84.