ON CLASSES OF HARMONIC FUNCTIONS OF CARLEMAN TYPE

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ABSTRACT. Let \( f \) be harmonic functions on the unit disk \( D \) of the complex plane \( \mathbb{C} \). We show that \( f \) can be expanded in a series \( f = \sum_n f_n \), where \( f_n \) is a harmonic function on \( D_n, \Gamma, A \) satisfying \( \sup_{z \in D_n, \Gamma, A} |f_n(z)| \leq C \rho^n \) for some constants \( C > 0 \) and \( 0 < \rho < 1 \), and where \( (D_n, \Gamma, A)_n \) is a suitably chosen sequence of decreasing neighborhoods of the closure of \( D \). Conversely, if \( f \) admits such an expansion then \( f \) is of Carleman type. The decrease of the sequence \( (D_n, \Gamma, A)_n \) characterizes the smoothness of \( f \). These constructions are perfectly explicit.

1. Introduction

It was shown for special classes of harmonic functions in [2] that Gevrey harmonic functions on the unit disk, \( D \), of the complex plane are, in fact, sums of certain series of harmonic functions on specific neighborhoods of \( \overline{D} \). It seems that a similar result holds in largest classes. This gives a motivation for asking about classes of harmonic functions of Carleman type. The principal difficulty is that we have to control an infinity of derivatives of a function. The purpose of this paper is to extend results of [2] to the case of Harmonic functions of Carleman type. Our principal result (Theorem 4.1) gives a more useful characterization. Precisely, we show that the Harmonic functions of Carleman type on \( D \) are exactly those which are sums of certain series in specific neighborhoods of \( \overline{D} \).

2. Notations and Definitions

Let \( D = \{z \in \mathbb{C}, |z| < 1\} \) the disk of the plane \( \mathbb{C} \) and \( m(t) \) be a real-valued \( C^\infty \) function defined for \( t \gg 0 \). We suppose that \( m(t), m'(t), m''(t) \) are strictly positive and \( \lim_{t \to \infty} m'(t) = +\infty \); we suppose also that there exists \( \delta > 0 \) such that \( m''(t) \leq \delta \). Put \( M(t) = \exp(m(t)) \). \( \mathcal{H}(D) \) denotes the spaces of harmonic functions on a neighborhood of \( \overline{D} \). Let us consider \( \mathcal{H}_M(D) \) to be the class of

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harmonic functions on \( \mathbb{D} \) given by those \( f \) for which there are positive constants \( C \) and \( \rho \) such that
\[
|f^{(n)}(z)| \leq Cp^n M(n),
\]
for any \( z = x + iy \in \mathbb{D} \) and any \( n = p + q \in \mathbb{N} \) where
\[
f^{(n)}(z) := \frac{\partial^n f}{\partial x^p \partial y^q}(z).
\]
Note that every function \( f \) belonging to a class \( \mathcal{H}_M(\mathbb{D}) \) can be extended in a unique way to a \( C^\infty \) function on \( \overline{\mathbb{D}} \): if \( \xi \in \partial \mathbb{D} \) and if \( z_j \rightarrow \xi \) all partial derivatives of \( f \) at \( z_j \) are uniformly bounded on \( \mathbb{D} \) and we can apply the mean value theorem. We denote this extension by the same symbol \( f \). The class \( \mathcal{H}(\mathbb{D}) \) correspond, then, to \( M(t) = t^i \), so that \( m(t) = t \ln(t) \). It is interesting to note that there exist functions in \( H_m(\mathbb{D}) \) which are monotonic at infinity, we suppose that \( M(t) = t \ln(t) + t \mu(t) \) with \( \mu(t) \) being a strictly increasing \( C^\infty \) function for \( t \gg 0 \). In this short paper, we will consider classes that contain strictly \( H(\mathbb{D}) \); for this end we suppose that \( m(t) = t \ln(t) + t \mu(t) \) with \( \mu(t) \) being a strictly increasing \( C^\infty \) function for \( t \gg 0 \)

In view of Cauchy’s inequalities, Stirling formula and Heine–Borel theorem, if \( f \) is harmonic on neighborhood of \( \mathbb{D} \) and real valued, then \( f \) is real analytic on \( \mathbb{D} \) and the restriction of \( f \) to \( \mathbb{D} \) belongs to the class \( \mathcal{H}_M(\mathbb{D}) \). The class \( \mathcal{H}(\mathbb{D}) \) is strictly increasing according to the same symbol \( \mu(t) \). For \( m(t) = t \ln(t) \), \( m(t) + a(t) \), \( a(t) \), \( m(t) \) being a strictly increasing \( C^\infty \) function for \( t \gg 0 \) and \( \mu(t) \) being a strictly increasing \( C^\infty \) function for \( t \gg 0 \).

It is interesting to note that there exist functions in \( \mathcal{H}_M(\mathbb{D}) \) which are not harmonic on any neighborhood of \( \mathbb{D} \). Take, for instance,
\[
f(z) = \sum_{p \in \mathbb{Z}} \exp \left( -\sqrt{|p|}|z|^p \exp(ip\theta) \right).
\]
This function belongs to \( \mathcal{H}_M(\mathbb{D}) \) for \( \mu(t) = \ln(t) \) but it cannot be extended to be harmonic on any neighborhood of \( \mathbb{D} \) as we verify easily [2, p. 413].

Finally the condition \( m''(t) \leq \delta \) means that \( m(t) \) has a growth at infinity less than \( t^2 \); then we may suppose also that
\[
(2.2) \quad \mu(t) \leq at, \ t \gg 0, \ a > 0.
\]

3. The associated functions \( \Omega(s) \) and \( \Gamma(u) \)

Set \( \Omega(s) := \inf_{t \geq t_0} s^{-t} M(t) \), \( s \gg 0 \), where \( t_0 \) is fixed. The infimum is attained when \( m'(t) = \ln(s) \). The function \( t m'(t) + \mu(t) \) tends to infinity as \( t \rightarrow \infty \) and so it is strictly increasing \( (\mu(t) \) belongs to a Hardy field); so we have a unique value
of $t$ where the infimum is attained. Thus, if $\Omega(s) = \exp(-\omega(s))$, then we get the system

$$s = \exp(m'(t)), \quad \omega(s) = t m'(t) - m(t). \tag{3.1}$$

Since $\mu'(t) > 0$, we have $\omega(s) > 0$ and $\lim_{s \to +\infty} \omega(s) = +\infty$. Thus $\Omega(s)$ is strictly decreasing and $\lim_{s \to +\infty} \Omega(s) = 0$. Set $\Gamma(u) := \exp(-\gamma(u))$, where $u$ and $\gamma(u)$ are defined by

$$u = t^2 \mu'(t), \quad \gamma(u) = t \mu'(t) + \mu(t). \tag{3.2}$$

as $\mu(t)$ is strictly increasing, and $\lim_{t \to +\infty} \mu(t) = +\infty$, it follows that $\gamma(u)$ is strictly increasing and $\lim_{u \to +\infty} \gamma(u) = +\infty$. Hence $\Gamma(u)$ is strictly decreasing and $\lim_{u \to +\infty} \Gamma(u) = 0$. System (3.2) gives easily

$$t = \frac{1}{\gamma'(u)}, \quad \mu(t) = \gamma(u) - u \gamma'(u), \tag{3.3}$$

which shows that $\gamma'(u)$ is strictly decreasing, positive and $\lim_{u \to +\infty} \gamma'(u) = 0$. Note that $\gamma(u)$, just as $\mu(t)$, is defined modulo an additive constant.

### 4. Main Result

By harmonic polynomial on $\mathbb{R}^2 \simeq \mathbb{C}$, we mean a complex polynomial $P$ of two variables which satisfies Laplace’s equation, $\Delta P = 0$. In other words, $P$ is a finite linear combinations, on the field $\mathbb{C}$, of harmonic polynomials $\delta_n$ ($n \geq 1)$:

$$\delta_1 = 1, \quad \delta_n(r \exp(i\theta)) = \begin{cases} r^k \cos k\theta & \text{if } n = 2k, \\ r^k \sin k\theta & \text{if } n = 2k + 1. \end{cases}$$

Consequently a harmonic polynomial is a polynomial in $|z|$ with coefficients in $\mathbb{C}$. Define $\mathbb{D}_{n,\Gamma,A} := \{ z \in \mathbb{C}; \; d(z, \mathbb{D}) < A \Gamma(n) \}$, where $A$ is a positive real number and $n = 1, 2, \ldots$.

Under the condition

$$\lim_{t \to +\infty} \frac{\ln(t)}{\mu(t)} \neq 0 \tag{4.1}$$

we prove the following results:

**Theorem 4.1.** (1) Let $f \in \mathcal{H}_M(\mathbb{D})$. Then there exist constants $C > 0$, and $\rho$ with $0 < \rho < 1$; there exist a sequence $(P_l)_l$ of harmonic polynomials defined on $\mathbb{D}_{l,\Gamma,1}$ such that $f(z) = \sum_{l \geq 0} P_l(z)$ and $\|P_l\|_{\mathbb{D}_{l,\Gamma,1}} \leq C \rho^l$, for every $l \gg 0$.

(ii) Conversely, suppose that there exist constants $A > 0$, $C > 0$, $0 < \rho < 1$, and a sequence $(f_n)_{n \geq 1}$ of harmonic functions on $(\mathbb{D}_{n,\Gamma,A})_n$, such that $\|f_n\|_{\mathbb{D}_{n,\Gamma,A}} \leq C \rho^n$ for all $n \gg 1$; then the series $\sum_n f_n := f$ belong to $\mathcal{H}_M(\mathbb{D})$. 

5. Proof of Theorem 4.4

Following [24], we denote by $\mathcal{H}_M([0, 2\pi])$ the indefinitely differentiable functions $g$ on the interval $[0, 2\pi]$, $g \in C^\infty([0, 2\pi])$, such that the following holds. There exist constants $C > 0$ and $\rho > 0$ such that $|g^n(x)| \leq C\rho^n M(n)$ for all $x \in [0, 2\pi]$ and for all $n \in \mathbb{N}$. By remarking a fact that a given function $f$ which is harmonic on $\mathbb{D}$ and $C^\infty$ on $\overline{\mathbb{D}}$ belongs to the class $\mathcal{H}_M(\mathbb{D})$ if and only if the function $\theta \mapsto g(\theta) := f(\exp(i\theta))$ belongs to the class $\mathcal{H}_M([0, 2\pi])$, we have the following:

Proposition 5.1. Let $f(z) = \sum_{p \in \mathbb{Z}} a_p |p|^{\omega} \exp(i p \theta)$ be a harmonic function on $\mathbb{D}$ and $C^\infty$ on $\overline{\mathbb{D}}$. Then $f \in \mathcal{H}_M(\mathbb{D})$ if and only if, there exist constants $C > 0$ and $\rho > 0$ such that $|a_p| \leq C e^{-\rho} \Omega(\frac{|p|}{\rho})$, $|p| \gg 0$.

Proof. If $f \in \mathcal{H}_M(\mathbb{D})$ then $g(\theta) := f(\exp(i\theta)) \in \mathcal{H}_M([0, 2\pi])$. There exist, then, constants $C > 0$ and $\rho > 0$ such that $|g^n(\theta)| \leq C\rho^n M(n)$. But $g^{(n)}(\theta) = \sum_{p \in \mathbb{Z}} a_p (i p)^n \exp(i p \theta)$; consequently $|a_p||p|^n \leq C\rho^n M(n)$ for every $n \geq 1$; and, so, $|a_p| \leq C \inf_{n \geq 1} (\frac{\rho}{|p|})^n M(n)$. A suitable application of Taylor’s formula shows that the last infimum is bounded by $\exp(\frac{\rho}{|p|}) M(t)$, which is equal to $e^{\frac{\rho}{\rho}} \Omega(\frac{|p|}{\rho})$. Conversely, if the coefficients $a_p$ satisfy these estimates then $|a_p| \leq C \inf_{n \geq 1} (\frac{\rho}{|p|})^n M(n)$; or equivalently $|a_p||p|^n \leq C \rho^n M(n)$, for every $n \geq 1$. Then, we have

$$|g^{(n)}(\theta)| \leq \sum_{p \in \mathbb{Z} \setminus \{0\}} |a_p||p|^n \frac{1}{|p|^2} \leq C \rho^{n+2} M(n+2) \sum_{p \in \mathbb{Z} \setminus \{0\}} \frac{1}{|p|^2}.$$ 

We conclude by (2.1) and the remark preceding the proposition (5.1) that $f \in \mathcal{H}_M(\mathbb{D})$. The proof is, then, complete. □

Let us remark that, in general, $f(z) = \sum_{p \in \mathbb{Z}} \Omega(|p|)|z|^p \exp(i p \theta)$. This function belongs to $\mathcal{H}_M(\mathbb{D})$ but it cannot be extended to be harmonic function on any neighborhood of $\mathbb{D}$ as we verify easily.

Proof of part 1 of Theorem 4.4. Let

$$f(z) = \sum_{p \in \mathbb{Z}} a_p |p|^\omega \exp(i p \theta) \in \mathcal{H}_M(\mathbb{D}).$$

Without loss of generality, by proposition (5.1) we can suppose that we have $|a_p| \leq C e^{-\omega(|p|)}$, $|p| \gg 0$. Let $n \geq 0$ and consider all indexes $p \in \mathbb{Z}$ satisfying

(5.1)

For $p$ satisfying (5.1) and $z \in \mathbb{D}_{n+1, \Gamma, 1}$ we have with $C_1 = C_0 e^{-\omega}$,

$$|a_p||p|^\omega \exp(i p \theta)| \leq C_1 e^{-\omega(|p|)|p| \ln(1 + \Gamma(n+1))} \leq C_1 e^{-\omega(|p|)+|p|\Gamma(n+1)} \leq C_1 e^{-\omega(|p|)+|p|\omega(|p|)} \leq C_1 e^{-\omega(|p|)+|p|\omega(|p|)}.$$
Using (3.1) and (3.2) and taking into account (5.1), we obtain
\[ |a_p| |z|^p \exp(ip\theta)| \leq C_1 e^{-u(p)} \leq C_1 e^{-n}. \]

Otherwise the number \( \delta_n \) of indexes \( p \) satisfying (5.1) is bounded by
\[ 2(u^{-1}(n+1) - u^{-1}(n)) = 2\left( \frac{1}{\Gamma'(n)} - \frac{1}{\Gamma'(n+1)} \right) = 2 \frac{\Gamma''(\theta)}{(\Gamma'(\theta))^2} \]
where \( n < \theta < n+1 \). Condition (2.2) implies that the function \( \Gamma \) has subexponential decay when \( u \to \infty \). Then the function \( \frac{1}{\Gamma'} \) and its derivative have also subexponential decay. Thus \( \delta_n \leq 2e^{\frac{1}{2}n} \) for \( n \to 0 \). Now, set
\[ P_\mu(z) = \sum_{u^{-1}(n) \leq |p| < u^{-1}(n+1)} a_p |z|^p \exp(ip\theta). \]

Clearly \( P_n \) is a harmonic polynomial and \( f(z) = \sum_{n \geq 0} P_n(z), \ z \in \mathbb{D} \). Furthermore, by the preceding estimates, we have \( \|P_n\|_{\mathbb{D}_{n+1}, \Gamma, A} \leq 2C_1 e^{\frac{1}{2}n} \). This completes the proof of part one of Theorem 4.1 by setting \( C = 2C_1 \) and \( \rho = e^{\frac{1}{2}} \). □

**Proof of part 2 of Theorem 4.1.** Without loss of generality, we can suppose that all \( f_n \) are real valued. Consequently \( f_n \) is the real part of holomorphic function \( g_n \) on \( \mathbb{D}_{n+1, \Gamma, A} \). We use Borel–Caratheodory’s inequality [5, p. 21], to get
\[ \|g_n\|_{\mathbb{D}_{n+1, \Gamma, A}} \leq 2 \left( \frac{A}{2} \Gamma(n) \right)^{-1} \left( 1 + \frac{A}{2} \Gamma(n) \right) \left\| f_n \right\|_{\mathbb{D}_{n+1, \Gamma, A}} + \left( \frac{A}{2} \Gamma(n) \right)^{-1} \left( 2 + \frac{3A}{2} \Gamma(n) \right) |f_n(0)|. \]

This implies
\[ \|g_n\|_{\mathbb{D}_{n+1, \Gamma, A}} \leq \frac{2C}{A} \left( 4 + \frac{5A}{2} \Gamma(n) \right) \frac{\rho^n}{\Gamma(n)} \quad \text{for } n \gg 0. \]

We obtain, with \( \lambda := -\ln(\rho) \),
\[ \|g_n\|_{\mathbb{D}_{n+1, \Gamma, A}} \leq \frac{2C}{A} \left( 4 + \frac{5A}{2} \Gamma(n) \right) \exp(-\lambda n + \gamma(n)) \quad \text{for } n \gg 0. \]

By the assumption (1.1) and by [11, p. 224], we can choose \( \gamma = \mu \) and hence there exists \( B > 0 \) such that \( \gamma(n) \leq B \ln(n) \) \( n \gg 0 \). Consequently,
\[ \|g_n\|_{\mathbb{D}_{n+1, \Gamma, A}} \leq C_1 \exp \left( -\frac{\lambda n}{2} \right) \quad \text{for } n \gg 0 \]
where \( C_1 = C(f, \Gamma, A) > 0 \) does not depend on \( n \). By Cauchy’s inequalities, for each \( p = 0, 1, \ldots \), we get
\[ \|g_n^{(p)}\|_\mathbb{D} \leq C_1 p! \left( \frac{2}{A} \right)^p \left( \Gamma(n) \right)^{-p} \exp \left( -\frac{\lambda n}{2} \right) \leq C_1 p! \left( \frac{2}{A} \right)^p \left( \max_{t \geq 0} \exp \left( -\frac{\lambda t}{4} + p\gamma(t) \right) \right) \exp \left( -\frac{\lambda n}{4} \right). \]
That is because the closed disk $\overline{D}(z, \frac{\Delta r(n)}{2})$ is contained in $D_{n+1}$ for every $z \in D$.

On the other hand the maximum of the function $u \mapsto \exp\left(-\frac{\lambda}{\mu} u + p \gamma(u)\right)$ is obtained at $a_0$ such that $\frac{\gamma(a_0)}{\mu} = \frac{2}{\lambda}$ and equal, by the system (3.3), to $\exp\left(p \mu\left(\frac{\lambda}{\mu}\right)\right)$.

By [1], Lemma 3, we can replace $\mu\left(\frac{\lambda}{\mu}\right)$ by $\mu(p)$. Then, we obtain
\[
\|g^{(p)}\|_D \leq C_1 p! \left(\frac{2}{\lambda}\right)^p \exp(p \mu(p)) \exp\left(-\frac{\lambda n}{4}\right).
\]
Adding the inequalities (5.2) over $n \gg 1$ and put $g := \sum_n g_n$, it follows that
\[
\|g^{(p)}\|_D \leq C_1 p! \left(\frac{2}{\lambda}\right)^p \exp(p \mu(p)) \sum_n \exp\left(-\frac{\lambda n}{4}\right);
\]
and, then, the derivatives of $f$ have similar estimates, i.e.,
\[
\|f^{(p)}\|_D \leq C_1 p! \left(\frac{2}{\lambda}\right)^p \exp(p \mu(p)) \sum_n \exp\left(-\frac{\lambda n}{4}\right).
\]
We conclude that $f \in \mathcal{H}_{\Delta r}(D)$. This finishes the proof of Theorem 4.1. □

Example 5.1. $\mu(t) = 1 \ln(t)$, $k > 0$, which correspond to Gevrey class of order $k$. From (3.2) we obtain $u = \frac{1}{k} t$ and $\gamma(u) = \frac{1}{k} \ln(t) + \frac{1}{k} = \frac{k}{k}(\ln(t) + \ln(k))$; so we can choose $\gamma(u) = \frac{1}{k} \ln(u)$. In this situation (that is, if $\mu(t) = \frac{1}{k} \ln(t)$, $k > 0$) Theorem 4.1 is exactly the result of [2].

Example 5.2. $\mu(t) = \beta \ln(\ln(t))$ $\beta > 0$. We obtain $u = \frac{\beta t}{\ln(t)}$, so $\ln(u) \sim \ln(t)$, and $\gamma(u) = \beta \ln(\ln(t)) + \frac{\beta}{\ln(t)}$; so we can choose $\gamma(u) = \beta \ln(\ln(u))$.

We can construct other examples by taking $\mu(t) = a_1 \ln_1(t) + \cdots + a_p \ln_p(t)$ where $\ln_1(t) = \ln(t)$ and $\ln_{p+1}(t) = \ln(\ln_p(t))$, $a_i$ are positive constants $i = 1, \ldots, p$ and $p \in \mathbb{N}^*$.

Example 5.3. $\mu(t) = at$, $a > 0$; (extremal case), we obtain $\gamma(u) = 2\sqrt{au}$.

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References