Subclasses of Meromorphic Functions Associated with a Convolution Operator

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Abstract. The purpose of the present paper is to introduce a subclass of meromorphic functions by using the convolution operator, that generalizes some well-known classes previously defined by different authors. We discussed inclusion results, radius problems, and some connections with a certain integral operator.

1. Introduction

Let \( H(U) \) be the class of functions analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \), and let \( \Sigma(p,n) \) denote the class of all meromorphic functions of the form
\[
f(z) = \frac{1}{z^p} + \sum_{j=n}^{\infty} a_j z^j, \quad z \in \dot{U} = U \setminus \{0\}, \quad (p, n \in \mathbb{N} = \{1, 2, 3, \ldots\}).
\] (1)

Let \( \mathcal{P}_k(\alpha) \) be the class of functions \( g \), analytic in \( U \), satisfying the condition \( g(0) = 1 \) and
\[
\int_0^{2\pi} \left| \frac{\text{Re} g(z) - \alpha}{1 - \alpha} \right| d\theta \leq k\pi,
\] (2)
where \( z = re^{i\theta}, \ 0 < r < 1, \ k \geq 2 \) and \( 0 \leq \alpha < 1 \). This class was introduced by Padmanabhan and Parvatham [15], and as a special case we note that the class \( \mathcal{P}_2(0) \) was introduced by Pinchuk [16]. Moreover, \( \mathcal{P}(\alpha) := \mathcal{P}_2(\alpha) \) is the class of analytic functions \( g \) in \( U \), with \( g(0) = 1 \), and the real part greater than \( \alpha \).

Remark 1.1. (i) Like in [13] and [14], from the definition (2) it can easily be seen that the function \( g \), analytic in \( U \), with \( g(0) = 1 \), belongs to \( \mathcal{P}_k(\alpha) \) if and only if there exists the functions \( g_1, g_2 \in \mathcal{P}(\alpha) \) such that
\[
g(z) = \left( \frac{k}{4} + \frac{1}{2} \right) g_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) g_2(z).
\] (3)

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(ii) Notice that, if \( g \in H(U) \) with \( g(0) = 1 \), then there exist functions \( g_1, g_2 \in H(U) \) with \( g_1(0) = g_2(0) = 1 \), such that the function \( g \) can be written in the form (3). For example, taking
\[
g_1(z) = \frac{g(z) - 1}{k} + \frac{g(z) + 1}{2} \quad \text{and} \quad g_2(z) = \frac{g(z) + 1}{2} - \frac{g(z) - 1}{k},
\]
then \( g_1, g_2 \in H(U) \), and \( g_1(0) = g_2(0) = 1 \).

(iii) Using the fact that \( \mathcal{P}(\alpha) \) is the class of functions with real part greater than \( \alpha \), from the above representation formula it follows that
\[
\mathcal{P}_k(\alpha_2) \subset \mathcal{P}_k(\alpha_1), \quad \text{if} \quad 0 \leq \alpha_1 < \alpha_2 < 1.
\]

(iv) It is well-known from [12] that the class \( \mathcal{P}_k(\alpha) \) is a convex set.

We recall the differential operator \( \mathcal{D}^m_{\lambda, p} : \Sigma(p, n) \to \Sigma(p, n) \), defined as follows:
\[
\mathcal{D}^m_{\lambda, p} f(z) = f(z),
\]
\[
\mathcal{D}^m_{\lambda, p} f(z) = (1 - \lambda) \mathcal{D}^{m-1}_{\lambda, p} f(z) + \lambda \frac{(z^{p+1} \mathcal{D}^{m-1}_{\lambda, p} f(z))'}{z^p} = \frac{1}{z^p} \sum_{j=0}^{\infty} [1 + \lambda(j + p)] z^j, \quad (\lambda \geq 0, \ p \in \mathbb{N}, \ m \in \mathbb{N}),
\]
where the function \( f \in \Sigma(p, n) \) is given by (1). This operator could be written by using the Hadamard (convolution) product, like
\[
\mathcal{D}^m_{\lambda, p} = q_{p,n}(\lambda, m; z) \ast f(z),
\]
where
\[
q_{p,n}(\lambda, m; z) = \frac{1}{z^p} + \sum_{j=0}^{\infty} [1 + \lambda(j + p)] z^j.
\]

From the expansion formula (4) it is easy to verify the differentiation relation
\[
\lambda z \left( \mathcal{D}^m_{\lambda, p} f(z) \right)' = \mathcal{D}^{m+1}_{\lambda, p} f(z) - (1 + \lambda p) \mathcal{D}^m_{\lambda, p} f(z).
\]

Remark 1.2. The operator \( \mathcal{D}^m_{\lambda, p} \) was defined and studied by Aouf et al. [2] and Aouf and Seoudy [3], and we note that:

(i) The operator \( \mathcal{D}^m_{1,p} = \mathcal{D}^m_{p} \) was introduced and studied by Aouf and Hossen [1], Liu and Owa [8], Liu and Srivastava [9], and Srivastava and Patel [20].

(ii) The operator \( \mathcal{D}^m_{1,1} = \mathcal{D}^m \) was introduced and studied by Uralegaddi and Somanatha [21]. More general results than the work [21], with a different notation for convolution (to distinguish from the analytic case) were obtained in [17].

Next, by using the convolution operator \( \mathcal{D}^m_{\lambda, p} \), we will introduce the subclass of \( p \)-valent Bazilević functions of \( \Sigma(p, n) \) as follows:

Definition 1.3. A function \( f \in \Sigma(p, n) \) is said to be in the class \( \Sigma_{\lambda, p}^m(\mu; \gamma, \alpha) \) if it satisfies the condition
\[
(1 - \gamma) \left( z^p \mathcal{D}^m_{\lambda, p} f(z) \right) + \gamma \mathcal{D}^{m+1}_{\lambda, p} f(z) = \mathcal{D}^{m+1}_{\lambda, p} f(z) \in \mathcal{P}_k(\alpha),
\]
\[
(k \geq 2, \ \gamma \geq 0, \ \mu > 0, \ 0 \leq \alpha < 1),
\]
where all the powers represent the principal branches, i.e. \( \log 1 = 0 \).
We need to remark that, since the left-hand side function from the above definition need to be analytic in $U$, we implicitly assumed that $D_{\lambda, p}^m f(z) \neq 0$ for all $z \in U$.

To prove our main results, the following lemma will be required in our investigation. We emphasize that slightly general situation than the above lemma is covered in [18], which might be useful to cover the case of nonlinear differential subordination.

**Lemma 1.4.** [19] If $g$ is an analytic function in $U$, with $g(0) = 1$, and if $\lambda_1$ is a complex number satisfying $\Re \lambda_1 \geq 0$, $\lambda_1 \neq 0$, then

$$\Re [g(z) + \lambda_1 zg'(z)] > \alpha, \ z \in U, \quad (0 \leq \alpha < 1)$$

implies

$$\Re g(z) > \beta, \ z \in U,$$

where $\beta$ is given by

$$\beta = \alpha + (1 - \alpha)(2\beta_1 - 1), \quad \beta_1 = \int_0^1 (1 + t\Re \lambda_1)^{-1} dt,$$  \hspace{1cm} (7)

and $\beta_1$ is an increasing function of $\Re \lambda_1$, and $\frac{1}{2} \leq \beta_1 < 1$. The estimate is sharp in the sense that the bound cannot be improved.

In this paper we investigate several properties of the class $\Sigma B_{k}^m(p; \gamma, \mu, \alpha)$ associated with the operator $D_{\lambda, p}^m$, like inclusion results, radius problems, and some connections with the generalized Bernardi–Libera–Livingston integral operator introduced in [6].

2. Main Results

Unless otherwise mentioned, we assume throughout this paper that $k \geq 2$, $\gamma \geq 0$, $\mu > 0$, $0 \leq \alpha < 1$, and all the powers represent the principal branches, i.e. $\log 1 = 0$.

**Theorem 2.1.** If $f \in \Sigma B_{k}^m(p; \lambda; \gamma, \mu, \alpha)$, then

$$(z^p D_{\lambda, p}^m f(z))^{\mu} \in \mathcal{P}_k(\beta),$$  \hspace{1cm} (8)

where $\beta$ is given by (7), with $\lambda_1 = \frac{\gamma \lambda}{\mu}$.

**Proof.** Since the implication is obvious for $\gamma = 0$, suppose that $\gamma > 0$. Let $f$ be an arbitrary function in $\Sigma B_{k}^m(p; \lambda; \gamma, \mu, \alpha)$, and denote

$$g(z) := (z^p D_{\lambda, p}^m f(z))^{\mu}. \hspace{1cm} (9)$$

It follows that $g$ is analytic in $U$, with $g(0) = 1$, and according to the part (ii) of Remarks 1.1 the function $g$ can be written in the form

$$g(z) = (k^4 + \frac{1}{2}) g_1(z) - (k^4 - \frac{1}{2}) g_2(z), \hspace{1cm} (10)$$

where $g_1$ and $g_2$ are analytic in $U$, with $g_1(z) = g_2(z) = 1$.

From the part (i) of Remarks 1.1 we have that $g \in \mathcal{P}_k(\beta)$, if and only if the function $g$ has the representation given by the above relation, where $g_1, g_2 \in \mathcal{P}(\alpha)$. Consequently, supposing that $g$ is of the form (10), we will prove that $g_1, g_2 \in \mathcal{P}(\alpha)$.
Using the differentiation formula (6) and the notation (9), after an elementary computation we obtain
\[(1 - \gamma) \left( s^\mu D^n_{\lambda,p} f(z) \right) + \gamma \left( s^\mu D^n_{\lambda,p} f(z) \right) = g(z) + \frac{\gamma \lambda}{\mu} zg'(z). \quad (11)\]

Now, using the representation formula (3), we have
\[g(z) + \frac{\gamma \lambda}{\mu} zg'(z) = \left( \frac{k + \frac{1}{2}}{4} \right) \left[ g_1(z) + \frac{\gamma \lambda}{\mu} zg'(z) \right] - \left( \frac{k - \frac{1}{2}}{4} \right) \left[ g_2(z) + \frac{\gamma \lambda}{\mu} zg'(z) \right]. \quad (12)\]

Since \( f \in \Sigma B_k^m(p, \lambda; \gamma, \mu, \alpha) \), from the relations (11) and (12) it follows that
\[g_i(z) + \frac{\gamma \lambda}{\mu} zg'(z) \in \mathcal{P}(\alpha), \quad i = 1, 2. \quad (13)\]

To prove our result we need to show that (13) implies \( g_i \in \mathcal{P}(\beta) \), \( i = 1, 2 \). Thus, the conditions (13) are equivalent to
\[\text{Re} \left[ g_i(z) + \lambda_1 zg'(z) \right] > \alpha, \quad z \in \mathbb{U},\]

with \( \lambda_1 = \frac{\gamma \lambda}{\mu} \). According to Lemma 1.4, it follows that \( g_i \in \mathcal{P}(\beta) \), where \( \beta \) is given by (7), with \( \lambda_1 = \frac{\gamma \lambda}{\mu} \). Thus, according to the part (i) of Remarks 1.1 and to the representation formula (3) we obtain the desired result. \( \square \)

**Theorem 2.2.** If \( 0 \leq \gamma_1 < \gamma_2 \), then
\[\Sigma B_k^m(p, \lambda; \gamma_2, \mu, \alpha) \subset \Sigma B_k^m(p, \lambda; \gamma_1, \mu, \alpha).\]

**Proof.** If we consider an arbitrary function \( f \in \Sigma B_k^m(p, \lambda; \gamma_2, \mu, \alpha) \), then \( \varphi_2 \in \mathcal{P}(\alpha) \), where
\[\varphi_2(z) := (1 - \gamma_2) \left( s^\mu D^n_{\lambda,p} f(z) \right) + \gamma_2 \left( s^\mu D^n_{\lambda,p} f(z) \right).\]

According to Theorem 2.1 we have
\[\varphi_1(z) := \left( s^\mu D^n_{\lambda,p} f(z) \right) \in \mathcal{P}(\beta),\]

where \( \beta \) is given by (7), with \( \lambda_1 = \frac{\gamma \lambda}{\mu} \). Since \( \beta = \alpha + (1 - \alpha)(2\beta_1 - 1) \) and \( \frac{1}{2} \leq \beta_1 < 1 \), it follows that \( \beta \geq \alpha \), and from the part (ii) of Remarks 1.1 we conclude that \( \mathcal{P}(\beta) \subset \mathcal{P}(\alpha) \), hence \( \varphi_1 \in \mathcal{P}(\alpha) \).

A simple computation shows that
\[(1 - \gamma_1) \left( s^\mu D^n_{\lambda,p} f(z) \right) + \gamma_1 \left( s^\mu D^n_{\lambda,p} f(z) \right) = \left( 1 - \frac{\gamma_1}{\gamma_2} \right) \varphi_1(z) + \frac{\gamma_1}{\gamma_2} \varphi_2(z). \quad (14)\]

Since the class \( \mathcal{P}(\alpha) \) is a convex set (see the part (iv) of Remarks 1.1), it follows that right-hand side of (14) belongs to \( \mathcal{P}(\alpha) \) for \( 0 \leq \gamma_1 < \gamma_2 \), which implies that \( f \in \Sigma B_k^m(p, \lambda; \gamma_1, \mu, \alpha) \). \( \square \)
Theorem 2.3. Let us define the integral operator \( J_{c,p} : \Sigma(p,n) \to \Sigma(p,n) \) by
\[
J_{c,p}f(z) = \frac{c + 1}{z^{c+1}} \int_0^z t^{-p} f(t) dt \quad (c > -1).
\] (15)

We will give a short proof that this operator is well-defined, as follows. If the function \( f \in \Sigma(p,n) \) is of the form (1), then the definition (15) can be written
\[
J_{c,p}f(z) = \frac{1}{z^{c+1}} \int_0^z t^{-p} (t^c f(t)) dt = \frac{1}{z^{c+1}} \int_0^z t^c \varphi(t) dt = \frac{c + 1}{z^{c+1}} J_{c,p} \varphi(z),
\]
where
\[
J_{c,p} \varphi(z) = \frac{1}{z^{c+1}} \int_0^z t^c \varphi(t) dt
\]
and
\[
\varphi(z) = z^p f(z) = 1 + \sum_{j=n}^\infty a_j z^{j+p}, \quad z \in U,
\] (16)
is analytic in \( U \). We see that integral operator \( J_{c,p} \) defined above is similar to that of Lemma 1.2c. of [11]. According to this lemma, it follows that \( J_{c,p} \) is an analytic integral operator for any function \( \varphi \) of the form (16) whenever \( \text{Re} \, c > -1 \), and \( J_{c,p}f \in \Sigma(p,n) \) has the form
\[
J_{c,p}f(z) = \frac{1}{z^{c}} + \sum_{j=0}^{\infty} \frac{a_j}{j + p + c} z^j, \quad z \in U.
\]

The operator \( J_{c,p} \) was introduced by Kumar and Shukla [6], connected with the Bernardi–Libera–Livingston integral operators (see [4], [7] and [10]).

**Theorem 2.3.** If \( f \in \Sigma(p,n) \), the integral operator \( J_{c,p} \) is given by (15), \( \gamma \geq 0 \) and \( \mu > 0 \), then
\[
(1 - \gamma) \left( z^p D_{\lambda,p}^\mu J_{c,p}f(z) \right) + \gamma z^p D_{\lambda,p}^\mu f(z) \left( z^p D_{\lambda,p}^\mu J_{c,p}f(z) \right)^{\mu-1} \in \mathcal{P}_k(a),
\]
implies that
\[
\left( z^p D_{\lambda,p}^\mu J_{c,p}f(z) \right)^{\mu} \in \mathcal{P}_k(\beta),
\]
where \( \beta \) is given by (7), with \( \lambda_1 = \frac{\gamma}{\mu(c+1)} \).

**Proof.** Like in the remark mentioned after the Definition 1.3, since the left-hand side function from the above definition need to be analytic in \( U \), we implicitly assumed that \( D_{\lambda,p}^\mu J_{c,p}f(z) \neq 0 \) for all \( z \in U \).

The implication is obvious for \( \gamma = 0 \), hence suppose that \( \gamma > 0 \). Differentiating the relation (15) we have
\[
z \left( J_{c,p}f(z) \right)' = (c + 1)f(z) - (c + p + 1)J_{c,p}f(z),
\]
and using the fact that \( D_{\lambda,p}^\mu \) and \( J_{c,p} \) commute, this implies
\[
z \left( D_{\lambda,p}^\mu J_{c,p}f(z) \right)' = (c + 1)D_{\lambda,p}^\mu f(z) - (c + p + 1)D_{\lambda,p}^\mu J_{c,p}f(z).
\] (17)
If we let
\[ g(z) := (z^p D^m_{λ, p, f}(z))^μ, \]
then by part (ii) of Remarks 1.1 the function \( g \) can be written in the form (10), where \( g_1 \) and \( g_2 \) are analytic in \( U \), with \( g_1(0) = g_2(0) = 1 \). According to the the part (i) of Remarks 1.1 we need to prove that \( g_1, g_2 \in \mathcal{P}(β) \).

Using (17), from the above relation we have
\[
(1 - γ)(z^p D^m_{λ, p, f}(z))^μ + γ z^p D^m_{λ, p, f}(z)(z^p D^m_{λ, p, f}(z))^{μ-1} =
\]
\[
g(z) + \frac{γ}{μ(c + 1)} zg_1(z) = \left(\frac{k}{4} + \frac{1}{2}\right) \left[ g_1(z) + \frac{γ}{μ(c + 1)} zg_1^1(z) \right] - \frac{k}{4 - 2} \left[ g_2(z) + \frac{γ}{μ(c + 1)} zg_2^1(z) \right] \in \mathcal{P}(α).
\]

Now, from the part (i) of Remarks 1.1 it follows that
\[
g_i(z) + \frac{γ}{μ(c + 1)} zg_i^1(z) \in \mathcal{P}(α), \quad i = 1, 2,
\]
and from Lemma 1.4 we conclude that \( g_i \in \mathcal{P}(β), \quad i = 1, 2, \) with \( β \) given by (7) and \( λ_1 = \frac{γ}{μ(c + 1)} \). □

The following result represents the converse of Theorem 2.1.

**Theorem 2.4.** If \( f \in Σ(p, n) \) such that \( \left(z^p D^m_{λ, p, f}(z)\right)^μ \in \mathcal{P}(α) \), then \( ρ^p f(ρz) \in ΣB^m(p, λ; γ, μ, α) \), with
\[
ρ = \min \left\{ \frac{-nγλ + \sqrt{μ^2 - n^2γ^2λ^2}}{μ}; r_0 \right\}
\]
where
\[
r_0 = \begin{cases} \min \{ r > 0 : φ(r) = 0 \}, & \text{if } \exists r > 0 : φ(r) = 0 \\ 1, & \text{if } \forall r > 0 : φ(r) = 0, \end{cases}
\]
and
\[
φ(r) = (2α - 1)r^2 + 2\left[2α - 1 - n(1 - α)\frac{γλ}{μ}\right]r + 1.
\]

**Proof.** For an arbitrary \( f \in Σ(p, n) \) such that \( \left(z^p D^m_{λ, p, f}(z)\right)^μ \in \mathcal{P}(α) \), let \( g \) be defined as in (9), i.e.
\[
\left(z^p D^m_{λ, p, f}(z)\right)^μ = g(z) \in \mathcal{P}(α).
\]

From the part (i) of Remarks 1.1 we have that (20) holds if and only if
\[
g(z) = \left(\frac{k}{4} + \frac{1}{2}\right) g_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) g_2(z),
\]
where \( g_1, g_2 \in \mathcal{P}(α) \).
Using the above representation formula, like in the proof of Theorem 2.1 we deduce that

\[
(1 - \gamma) \left( z^p \mathcal{D}_{\alpha, p, f(z)}^{m+1} \right) + \gamma \frac{\mathcal{D}_{\alpha, p, f(z)}^{m+1}}{z^p \mathcal{D}_{\alpha, p, f(z)}^m} \left( z^p \mathcal{D}_{\alpha, p, f(z)}^m \right)^\mu = \left( \frac{k}{4} + \frac{1}{2} \right) \left[ g_1(z) + \frac{\gamma \lambda}{\mu} z g_1'(z) \right] - \left( \frac{k}{4} - \frac{1}{2} \right) \left[ g_2(z) + \frac{\gamma \lambda}{\mu} z g_2'(z) \right],
\]

and substituting \( G_i(z) := \frac{g_i(z) - \alpha}{1 - \alpha}, i = 1, 2 \), we finally obtain

\[
(1 - \gamma) \left( z^p \mathcal{D}_{\alpha, p, f(z)}^{m+1} \right) + \gamma \frac{\mathcal{D}_{\alpha, p, f(z)}^{m+1}}{z^p \mathcal{D}_{\alpha, p, f(z)}^m} \left( z^p \mathcal{D}_{\alpha, p, f(z)}^m \right)^\mu = \left( \frac{k}{4} + \frac{1}{2} \right) \left[ (1 - \alpha) \left( G_1(z) + \frac{\alpha}{1 - \alpha} + \frac{\gamma \lambda}{\mu} z G_1'(z) \right) - \frac{1}{2} \right] - \left( \frac{k}{4} - \frac{1}{2} \right) \left[ (1 - \alpha) \left( G_2(z) + \frac{\alpha}{1 - \alpha} + \frac{\gamma \lambda}{\mu} z G_2'(z) \right) \right],
\]

where \( G_1, G_2 \in \mathcal{P}(0) \).

To prove our result we need to determine the value of \( \rho \), such that

\[
\text{Re} \left[ G_i(z) + \frac{\alpha}{1 - \alpha} + \frac{\gamma \lambda}{\mu} z G_i'(z) \right] > 0, \quad \text{for } |z| < \rho, \ i = 1, 2,
\]

whenever \( G_1, G_2 \in \mathcal{P}(0) \).

Since \( f \in \Sigma(p, n) \), using the well-known estimates [5] for the class \( \mathcal{P}(0) \), i.e.

\[
|z G_i'(z)| \leq \frac{2m^n}{1 - r^{2n}} \text{Re} G_i(z), \quad |z| \leq r < 1, \ i = 1, 2,
\]

\[
\text{Re} G_i(z) \geq \frac{1 - r^n}{1 + r^n}, \quad |z| \leq r < 1, \ i = 1, 2,
\]

we conclude that

\[
\text{Re} \left[ G_i(z) + \frac{\alpha}{1 - \alpha} + \frac{\gamma \lambda}{\mu} z G_i'(z) \right] \geq \frac{\alpha}{1 - \alpha} + \text{Re} G_i(z) - \frac{\gamma \lambda}{\mu} |z G_i'(z)| \geq \frac{\alpha}{1 - \alpha} + \text{Re} G_i(z) \left[ 1 - \frac{\gamma \lambda}{\mu} \frac{2m^n}{1 - r^{2n}} \right],
\]

for all \( |z| \leq r < 1 \) and \( i = 1, 2 \).

A simple calculation shows that \( 1 - \frac{\gamma \lambda}{\mu} \frac{2m^n}{1 - r^{2n}} \geq 0 \) if and only if

\[
r \in \left[ 0, \left( -\frac{2\gamma \lambda}{\mu} + \frac{\mu^2 + n^2\gamma^2 \lambda^2}{\mu} \right) \right] \frac{1}{n},
\]

and assuming that (22) holds, from (21) we obtain

\[
\text{Re} \left[ G_i(z) + \frac{\alpha}{1 - \alpha} + \frac{\gamma \lambda}{\mu} z G_i'(z) \right] \geq \frac{\alpha}{1 - \alpha} + \frac{1 - r^n}{1 + r^n} \left[ 1 - \frac{\gamma \lambda}{\mu} \frac{2m^n}{1 - r^{2n}} \right], \quad |z| \leq r < 1, \text{ for } i = 1, 2.
\]
It is easy to check that the right-hand side of the above inequality is greater or equal than zero if and only if
\[ r \in [0, \min \{1; r_0\}], \]
where \( r_0 \) is given by (19), and combining this with (22) we obtain our result.

**Remark 2.5.** (i) For the special case \( n = 1 \), it follows that if \( f \in \Sigma(p, 1) \) then
\[
\rho = \min \left\{ -\frac{\gamma \lambda}{\mu} + \sqrt{\frac{\mu^2 + \gamma^2 \lambda^2}{\mu}}; r_0 \right\}.
\]
(ii) We remark that for the special case \( n = 1 \) and \( \alpha = 0 \), the formula (18) reduces to
\[
\rho = -\left(1 + \frac{\gamma \lambda}{\mu}\right) + \sqrt{\left(1 + \frac{\gamma \lambda}{\mu}\right)^2 + 1}.
\]
(iii) Putting \( \lambda = 1 \) in the above results, we obtain the similar results associated with the operator \( \mathcal{D}_m^\rho \).
(iv) Taking \( \lambda = p = 1 \) in the above results, we obtain the similar results involving the operator \( \mathcal{D}_m^\rho \).

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**References**


