On the Geometry of Trans-Para-Sasakian Manifolds

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Abstract. In this paper, we introduce the trans-para-Sasakian manifolds and we study their geometry. These manifolds are an analogue of the trans-Sasakian manifolds in the Riemannian geometry. We shall investigate many curvature properties of these manifolds and we shall give many conditions under which the manifolds are either $\eta$–Einstein or Einstein manifolds.

1. Introduction

In Grey-Hervella classification of almost Hermitian manifolds (see [3]), there appears a class, $W_4$, of Hermitian manifolds which are closely related to locally conformal Kähler manifolds. An almost contact structure on a manifold $M$ is called a trans-Sasakian structure (see [8]) if the product manifold $M \times \mathbb{R}$ belongs to the class $W_4$. The class $C_5 \oplus C_5$ (see [6], [7]) coincides with the class of trans-Sasakian structures of type $(\alpha, \beta)$. In fact, in (see [7]), local nature of the two subclasses, namely the $C_5$ and the $C_6$ structures, of trans-Sasakian structures are characterized completely. We note that the trans-Sasakian structures of type $(0,0), (0, \beta)$ and $(\alpha,0)$ are cosymplectic (see [1]), $\beta$–Kenmotsu (see [4]) and $\alpha$–Sasakian (see [4]), respectively. We consider the trans-para-Sasakian manifolds as an analogue of the trans-Sasakian manifolds. A trans-para-Sasakian manifold is a trans-para-Sasakian structure of type $(\alpha, \beta)$, where $\alpha$ and $\beta$ are smooth functions. The trans-para-Sasakian manifolds of types $(\alpha, \beta)$, and are respectively the para-cosymplectic, para-Sasakian (in case $\alpha = 1$, these are just the para-Sasakian manifolds; in case $\alpha = -1$, these are the quasi-para-Sasakian manifolds, see [11]) and para-Kenmotsu (for the case $\beta = 1$ see [12]). In the second section, we give the formal definition of trans-para-Sasakian manifolds of type $(\alpha, \beta)$ and we prove some basic properties. We give an example for a 3-dimensional trans-para-Sasakian manifold. In the last section, we investigate the curvature properties of the trans-para-Sasakian manifolds. Further, we find many conditions under which the manifolds are either $\eta$–Einstein or Einstein manifolds.
2. Preliminaries

A \((2n+1)\)-dimensional smooth manifold \(M^{(2n+1)}\) has an almost paracontact structure \((\varphi, \xi, \eta)\) if it admits a tensor field \(\varphi\) of type \((1, 1)\), a vector field \(\xi\) and a 1-form \(\eta\) satisfying the following compatibility conditions

\[
\begin{align*}
(i) & \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \\
(ii) & \quad \eta(\xi) = 1 \quad \varphi^2 = id - \eta \otimes \xi, \\
(iii) & \quad \text{distribution } \mathcal{D} : p \in M \rightarrow \mathcal{D}_p \subset T_pM : \\
& \quad \mathcal{D}_p = \text{Ker} \eta = \{X \in T_pM : \eta(X) = 0\} \text{ is called paracontact distribution generated by } \eta.
\end{align*}
\]

The tensor field \(\varphi\) induces an almost paracomplex structure [5] on each fibre on \(\mathcal{D}\) and \((\mathcal{D}, \varphi, g_\mathcal{D})\) is a \(2n\)-dimensional almost paracomplex distribution. Since \(g\) is non-degenerate metric on \(M\) and \(\xi\) is non-isotropic, the paracontact distribution \(\mathcal{D}\) is non-degenerate.

An immediate consequence of the definition of the almost paracontact structure is that the endomorphism \(\varphi\) has rank \(2n\), \(\varphi \xi = 0\) and \(\eta \circ \varphi = 0\), (see [1, 2] for the almost contact case).

If a manifold \(M^{(2n+1)}\) with \((\varphi, \xi, \eta)\)-structure admits a pseudo-Riemannian metric \(g\) such that

\[
g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y),
\]

then we say that \(M^{(2n+1)}\) has an almost paracontact metric structure and \(g\) is called compatible. Any compatible metric \(g\) with a given almost paracontact structure is necessarily of signature \((n + 1, n)\).

Note that setting \(Y = \xi\), we have \(\eta(X) = g(X, \xi)\).

Further, any almost paracontact structure admits a compatible metric.

**Definition 2.1.** If \(g(X, \varphi Y) = d\eta(X, Y)\) (where \(d\eta(X, Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X, Y]))\) then \(\eta\) is a paracontact form and the almost paracontact metric manifold \((M, \varphi, \xi, \eta, g)\) is said to be a paracontact metric manifold.

A paracontact metric manifold for which \(\xi\) is Killing is called a K-paracontact manifold. A paracontact structure on \(M^{(2n+1)}\) naturally gives rise to an almost paracomplex structure on the product \(M^{(2n+1)} \times \mathbb{R}\). If this almost paracomplex structure is integrable, then the given paracontact metric manifold is said to be para-Sasakian. Equivalently, (see [10]) a paracontact metric manifold is a para-Sasakian if and only if

\[
(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X,
\]

for all vector fields \(X\) and \(Y\) (where \(V\) is the Livi-Civita connection of \(g\)).

**Definition 2.2.** If \((\nabla_X \varphi)Y = \alpha(-g(X, Y)\xi + \eta(Y)\xi) + \beta(g(X, \varphi Y)\xi + \eta(Y)\varphi X),\) then the manifold \((M^{(2n+1)}, \varphi, \eta, \xi, g)\) is said to be a trans-para-Sasakian manifold.

From Definition 2.2 we have

\[
\nabla_X \xi = -\alpha \varphi X - \beta(X - \eta(X)\xi).
\]

**Definition 2.3.** A \((2n+1)\)-dimensional almost paracontact metric manifold is called normal if \(N(X, Y) - 2d\eta(X, Y)\xi = 0\), where \(N(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[X, \varphi Y]\) is the Nijenhuis torsion tensor of \(\varphi\) (see [10]).

Denoting by \(\mathcal{L}\) the Lie differentiation of \(g\), we see

**Proposition 2.4.** Let \((M^{(2n+1)}, \varphi, \eta, \xi, g)\) be a trans-para-Sasakian manifold. Then we have

\[
(\nabla_X \eta)Y = \alpha g(X, \varphi Y) - \beta(g(X, Y) - \eta(X)\eta(Y)),
\]

\[
d\eta(X, Y) = \alpha g(X, \varphi Y),
\]
the standard coordinates in $\nabla$ of $1$

Let $3$. Some curvature properties of trans-para-Sasakian manifolds

Finally, the sectional curvature $\nabla$

Let us consider the 3-dimensional manifold $M$

Example 2.5. Let us consider the 3-dimensional manifold $M^3 = \{ (x, y, z) : (x, y, z) \in \mathbb{R}^3 \}, z \neq 0$, where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$. We choose the vector fields

$E_1 = e^x \left( \frac{\partial}{\partial X} + y \frac{\partial}{\partial Z} \right), \quad E_2 = e^x \frac{\partial}{\partial Y}, \quad E_3 = \frac{\partial}{\partial Z}$

which are linearly independent at each point of $M$. We define an almost paracontact structure $(\varphi, \xi, \eta)$ and a pseudo-Riemannian metric $g$ in the following way:

$\varphi E_1 = E_2, \quad \varphi E_2 = E_1, \quad \varphi E_3 = 0$

$\xi = E_3, \quad \eta(E_3) = 1, \quad \eta(E_1) = \eta(E_2) = 0,$

$g(E_1, E_1) = g(E_3, E_3) = -g(E_2, E_2) = 1,$

$g(E_i, E_j) = 0, \quad i \neq j \in \{1, 2, 3\}$.

By the definition of Lie bracket, we have

$[E_1, E_2] = ye^2E_2 - e^2E_3, \quad [E_2, E_3] = -E_2, \quad [E_1, E_3] = -E_3.$

Then $(M, \varphi, \xi, \eta, g)$ is a 3-dimensional almost paracontact manifold. The Koszul equality becomes

$V_{E_1} E_1 = E_3, \quad V_{E_1} E_2 = -\frac{1}{2} e^{2z} E_3, \quad V_{E_1} E_3 = -E_1 - \frac{1}{2} e^{2z} E_2,$

$V_{E_2} E_1 = -ye^2E_2 + \frac{1}{2} e^{2z} E_3, \quad V_{E_2} E_2 = -ye^2E_1 - E_3, \quad V_{E_2} E_3 = -\frac{1}{2} e^{2z} E_1 - E_2,$

$V_{E_3} E_1 = -\frac{1}{2} e^{2z} E_2, \quad V_{E_3} E_2 = -\frac{1}{2} e^{2z} E_1, \quad V_{E_3} E_3 = 0.$

We have $V_{E_i} \xi = -\alpha \varphi E_i - \beta \eta \varphi E_i, \quad V_{E_i} \xi = -\alpha \varphi E_i - \beta \eta \varphi E_i, \quad V_{E_i} \xi = 0$ for $E_3 = \xi$, where $\alpha = \frac{1}{2} e^{2z}$ and $\beta = 1$.

Again, by virtue of (5) and $(V_{E_i} \eta) Y = Y(\eta(\varphi Y)) - \eta(V_{E_i} Y)$ we obtain

$(V_{E_i} \eta) E_1 = -\beta = -1, \quad (V_{E_i} \eta) E_1 = \alpha = -\frac{1}{2} e^{2z}, \quad (V_{E_i} \eta) E_1 = 0.$

Thus from above the calculation the condition (4) and (5) are satisfied and the structure $(\varphi, \xi, \eta, g)$ is a trans-para-Sasakian structure of type $(\alpha, \beta)$, where $\alpha = \frac{1}{2} e^{2z}$ and $\beta = 1$. Consequently $(M^3, \varphi, \xi, \eta, g)$ is a trans-para-Sasakian manifold.

Finally, the sectional curvature $K(\xi, X) = \epsilon_X R(X, \xi, \xi, X)$, where $|X| = \epsilon_X = \pm 1$, of a plane section spanned by $\xi$ and the vector $X$ orthogonal to $\xi$ is called $\xi$-sectional curvature, where denoting by $R$ the curvature tensor of $V$.

3. Some curvature properties of trans-para-Sasakian manifolds

We begin with the following Lemma.

Lemma 3.1. Let $(M^{2n+1}, \varphi, \eta, \xi, g)$ be a trans-para-Sasakian manifold. Then we have

$R(X, Y) \xi = -\alpha^2 + \beta^2)(\eta(Y)X - \eta(X)Y) - 2\alpha \beta(\eta(Y)\varphi X - \eta(X)\varphi Y) - X(\alpha)\varphi Y + Y(\alpha)\varphi X + Y(\beta)\varphi^2 X - X(\beta)\varphi^2 Y$. (10)
Proof. Using Definition 2.2, we obtain
\[ \nabla_X \nabla_Y \xi = \nabla_X (-\alpha \varphi Y - \beta (Y - \eta(Y))\xi) = \]
\[ = -X(\alpha)\varphi Y - aV_X \varphi Y - X(\beta)\varphi^2 Y - \beta V_X Y - \beta (X\eta(Y))\xi - \]
\[ -\alpha \beta \eta(Y)\varphi X - \beta^2 \eta(Y)X + \beta^2 \eta(X)\eta(Y)\xi, \]

From here and (4), we get
\[ R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]} \xi = \]
\[ = -X(\alpha)\varphi Y + Y(\alpha)\varphi X - a((V_X \varphi)Y - (V_Y \varphi)X) - \]
\[ -X(\beta)\varphi^2 Y + Y(\beta)\varphi^2 X + \beta((V_X \eta)Y - (V_Y \eta)X)\xi - \]
\[ -\alpha \beta (\eta(Y)\varphi X - \eta(X)\varphi Y) - \beta^2 (\eta(Y)X - \eta(X)Y), \]

which in view of Definition 2.2 and (5) gives (10). \( \square \)

Lemma 3.1 yields the following

Proposition 3.2. If \((M^{2n+1}, \varphi, \eta, \xi, g)\) is a trans-para-Sasakian manifold, then it is of \( \xi \)-sectional curvature \( K(\xi, X) = -\varepsilon_X(\alpha^2 + \beta^2 - \xi(\beta)). \)

In a trans-para-Sasakian manifolds the functions \( \alpha \) and \( \beta \) can not be arbitrary. This fact is shown in the following

Theorem 3.3. In trans-para-Sasakian manifold, we have
\[ R(\xi, X)\xi = (\alpha^2 + \beta^2 - \xi(\beta))(X - \eta(X))\xi, \quad (11) \]
\[ 2\alpha \beta - \xi(\alpha) = 0. \quad (12) \]

Proof. Using (10) in \( R(\xi, Z, X, Y) = R(X, Y, \xi, Z) \), we get
\[ R(\xi, Z)X = - (\alpha^2 + \beta^2)(g(X, Z) - \eta(X)Z) - 2\alpha \beta (g(\varphi X, Z)\xi + \eta(X)\varphi Z) + \]
\[ +X(\alpha)\varphi Z + g(\varphi X, Z)\varphi \alpha - X(\beta)(Z - \eta(Z))\xi - \eta(\varphi X, \varphi Z)\varphi \beta. \quad (13) \]

From (10), we get
\[ R(\xi, X)\xi = (\alpha^2 + \beta^2 - \xi(\beta))(X - \eta(X))\xi + (2\alpha \beta - \xi(\alpha))\varphi Y, \]

while gives us (10)
\[ R(\xi, X)\xi = (\alpha^2 + \beta^2 - \xi(\beta))(X - \eta(X))\xi - (2\alpha \beta - \xi(\alpha))\varphi Y. \]

The above two equations provide (11) and (12). \( \square \)

From Lemma 3.1, we have the following

Proposition 3.4. In a \((2n + 1)\)-dimensional tran-para-Sasakian manifold, we have
\[ \text{Ric}(X, \xi) = -(2n(\alpha^2 + \beta^2) - \xi(\beta))\eta(X) + (2n - 1)\xi(\beta) - \varphi X(\alpha), \quad (14) \]
\[ Q\xi = -(2n(\alpha^2 + \beta^2) - \xi(\beta))\xi + (2n - 1)\varphi \alpha + \varphi \eta(\varphi \alpha), \quad (15) \]

where \( \text{Ric} \) is the Ricci tensor and \( Q \) is the Ricci operator given by
\[ \text{Ric}(X, Y) = g(QX, Y). \quad (16) \]
Corollary 3.5. If in a \((2n + 1)\)-dimensional trans-para-Sasakian manifold we have \(\varphi(\text{grad}a) = -(2n - 1)\text{grad} \beta\), then
\[
\xi(\beta) = g(\xi, \text{grad} \beta) = -\frac{1}{2n - 1} g(\xi, \varphi(\text{grad}a)) = 0,
\]
and hence
\[
\text{Ric}(X, \xi) = -2n(\alpha^2 + \beta^2)\eta(X),
\]
\[
Q\xi = -2n(\alpha^2 + \beta^2) \xi.
\]

From here on, we shall assume that \(\varphi(\text{grad}a) = -(2n - 1)\text{grad} \beta\).

The Weyl-projective curvature tensor \(P\) is defined as
\[
P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}(\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y).
\]

Hence we can state the following

Theorem 3.6. A Weyl projectively flat trans-para-Sasakian manifold is an Einstein manifold.

Proof. Suppose that \(P = 0\). Then from equation (19), we have
\[
R(X, Y)Z = \frac{1}{2n}(\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y).
\]

From (20), we obtain
\[
R(X, Y, Z, W) = \frac{1}{2n}(\text{Ric}(Y, Z)\eta(X) - \text{Ric}(X, Z)\eta(Y)).
\]

Putting \(W = \xi\) in (21), we get
\[
\eta(R(X, Y)Z) = \frac{1}{2n}(\text{Ric}(Y, Z)\eta(X) - \text{Ric}(X, Z)\eta(Y)).
\]

Again taking \(X = \xi\), and using (10) and (17), we get
\[
\text{Ric}(X, Y) = -2n(\alpha^2 + \beta^2)\eta(X, Y).
\]

\(\square\)

Theorem 3.7. A trans-para-Sasakian manifold satisfying \(R(X, Y)P = 0\) is an Einstein manifold and also it is a manifold of scalar curvature \(\text{scal} = -2n(2n + 1)(\alpha^2 + \beta^2)\).

Proof. Using (10) and (17) in (19), we get
\[
\eta(P(X, Y)\xi) = 0
\]
and
\[
\eta(P(\xi, Y)Z) = -(\alpha^2 + \beta^2)\eta(Y, Z) - \frac{1}{2n}\text{Ric}(Y, Z)
\]

Now,
\[
\]

By assumption \(R(X, Y)P = 0\), so we have
\[
\]
Therefore
\[ g(R(\xi, Y)P(U, V)Z, \xi) - g(P(R(\xi, Y)U, V)Z, \xi) - g(P(\xi, R(\xi, Y)V)Z, \xi) - g(P(U, V)R(\xi, Y))Z, \xi) = 0. \]

From this, it follows that,
\[ -P(U, V, Z, Y) + \eta(Y)\eta(P(U, V)Z) - \eta(U)\eta(P(Y, V)Z) + g(Y, U)\eta(P(\xi, V)Z) - g(Y, V)\eta(P(U, \xi)Z) + \eta(Z)\eta(P(U, V)Y) = 0. \]  
Let \( e_i \), \( i = 1, ..., 2n + 1 \) be an orthonormal basis. Then summing up for \( 1 \leq i \leq 2n + 1 \) of the relation (27) for \( Y = U = e_i \) yields
\[ 2n\eta(P(\xi, V)Z) + \eta(Z)P(V, e_i, e_i, \xi) = 0. \]  
From (25), we have
\[ Ric(V, Z) = -2n(\alpha^2 + \beta^2)g(Y, Z) - ((2n + 1)(\alpha^2 + \beta^2) + \frac{scal}{2n}). \]  
Taking \( Z = \xi \) in (29) and using (17) we obtain
\[ scal = -2n(2n + 1)(\alpha^2 + \beta^2) \text{ and } Ric(V, Z) = -2n(\alpha^2 + \beta^2)g(Y, Z) \]  

\[ \square \]

The Weyl-conformal tensor \( C \) is defined by
\[ C(X, Y)Z = R(X, Y)Z - \frac{1}{2n - 1}(g(Y, Z)QX - g(X, Z)QY + Ric(Y, Z)X - \text{scal}(2n - 1)(g(Y, Z)X - g(X, Z)Y). \]  
We have the following

**Theorem 3.8.** A conformally flat trans-para-Sasakian manifold is an \( \eta \)-Einstein manifold.

**Proof.** Suppose that \( C = 0 \). Then from (31), we get
\[ R(X, Y)Z = \frac{1}{2n - 1}(g(Y, Z)QX - g(X, Z)QY + Ric(Y, Z)X - \text{scal}(2n - 1)(g(Y, Z)X - g(X, Z)Y). \]  
From the identity (32), we have
\[ \eta(R(X, Y)Z) = \frac{1}{2n - 1}(g(Y, Z)Ric(X, \xi) - g(X, Z)Ric(Y, \xi) + \eta(X)Ric(Y, Z) - \text{scal}(2n - 1)(g(Y, Z)\eta(X) - g(X, Z)\eta(Y). \]  
Again taking \( X = \xi \) in (33), and using (10) and (17) we get
\[ Ric(X, Y) = ((\alpha^2 + \beta^2) + \frac{scal}{2n})g(Y, Z) - ((2n + 1)(\alpha^2 + \beta^2) + \frac{scal}{2n})\eta(Y). \]  

\[ \square \]
Theorem 3.9. A trans-para-Sasakian manifold satisfying \( R(X,Y)C = 0 \) is an \( \eta \)-Einstein manifold.

Proof. From identity (31), we have \( \eta(C(X,Y)\xi) = 0 \) and

\[
\eta(C(\xi, Y)Z) = \frac{1}{2n-1}((\alpha^2 + \beta^2) + \frac{\text{scal}}{2n})(g(Y, Z) - \eta(Y)\eta(Z)) - \frac{1}{2n-1}(\text{Ric}(Y, Z) + 2n(\alpha^2 + \beta^2)\eta(Y)\eta(Z)). \tag{35}
\]

Now,

\[
(R(X,Y)C(U,V)Z = R(X,Y)(C(U,V)Z) - C(R(X,Y)U,V)Z - C(U, R(X,Y)V)Z - C(U, V)R(X,Y)Z.
\]

By assumption \( R(X,Y)C = 0 \), so we have

\[
R(X,Y)C(U,V)Z - C(R(X,Y)U,V)Z - C(U, R(X,Y)V)Z - C(U, V)R(X,Y)Z = 0. \tag{36}
\]

Therefore

\[
g(R(\xi, Y)C(U,V)Z, \xi) - g(C(R(\xi, Y)U,V)Z, \xi) - g(C(U, R(\xi, Y)V)Z, \xi) - g(C(U, V)R(\xi, Y)Z, \xi) = 0.
\]

From this, it follows that,

\[
-C(U, V, Z, Y) + \eta(Y)\eta(C(U, V)Z) - \eta(U)\eta(C(Y, V)Z) + \eta(V)\eta(C(U, \xi)Z) \tag{37}
\]

is an Einstein manifold.

Let \( \{e_i\}, i = 1, ..., 2n + 1 \) be an orthonormal basis. Then summing up for \( 1 \leq i \leq 2n + 1 \) of the relation (37) for \( Y = U = e_i \) yields

\[
\eta(C(\xi, Y)Z) = 0. \tag{38}
\]

From (35), we have

\[
\text{Ric}(Y, Z) = \left(\frac{\text{scal}}{2n} + (\alpha^2 + \beta^2)\right)g(Y, Z) - ((2n + 1)(\alpha^2 + \beta^2) + \frac{\text{scal}}{2n})\eta(Y)\eta(Z). \tag{39}
\]

The concicular curvature tensor \( \overline{C} \) is defined by

\[
\overline{C}(X, Y)Z = R(X, Y)Z - \frac{\text{scal}}{2n(2n + 1)}(g(Y, Z)X - g(X, Z)Y). \tag{40}
\]

We have the following

Theorem 3.10. A trans-para-Sasakian manifold satisfying \( R(X,Y)\overline{C} = 0 \) is an Einstein manifold and a manifold of scalar curvature \( \text{scal} = -2n(2n - 1)(\alpha^2 + \beta^2) \).

Proof. From equality (40), we have \( \eta(\overline{C}(X,Y)\xi) = 0 \) and

\[
\eta(\overline{C}(\xi, Y)Z) = (-\frac{\text{scal}}{2n(2n + 1)} + (\alpha^2 + \beta^2))(g(Y, Z) - \eta(Y)\eta(Z)). \tag{41}
\]

Now,

\[
(R(X,Y)\overline{C}(U,V)Z = R(X,Y)\overline{C}(U,V)Z - \overline{C}(R(X,Y)U,V)Z - \overline{C}(U, R(X,Y)V)Z - \overline{C}(U, V)R(X,Y)Z.
\]

By assumption \( R(X,Y)\overline{C} = 0 \), so we have

\[
R(X,Y)\overline{C}(U,V)Z - \overline{C}(R(X,Y)U,V)Z - \overline{C}(U, R(X,Y)V)Z - \overline{C}(U, V)R(X,Y)Z = 0. \tag{42}
\]
Theorem 3.11. A trans-para-Sasakian manifold satisfying $R(X, Y)\Pi = 0$ is an Einstein manifold and a manifold of scalar curvature $\text{scal} = -2n(2n - 1)(\alpha^2 + \beta^2)$.

Proof. From the identity $R(X, Y)\Pi = 0$, we get

$$\Pi(R(X, Y)U, V) + \Pi(U, R(X, Y)V) = 0.$$  (47)

Putting $X = U = \xi$ and using (10) and (47) we have

$$-(\alpha^2 + \beta^2)\eta(Y)\Pi(\xi, V) + g(Y, V)\Pi(\xi, V) - \eta(V)\Pi(\xi, Y) = 0.$$  (48)

Using (47) in (48), we obtain that $Ric(X, Y) = -2n(\alpha^2 + \beta^2)g(X, Y)$ and $\text{scal} = 2n(2n - 1)(\alpha^2 + \beta^2)$. \square

The pseudo-projective curvature tensor is defined by

$$\Pi(X, Y) = \frac{(2n + 1)}{2n} Ric(X, Y) - \frac{\text{scal}}{2n} g(X, Y).$$  (46)

We have the following

**Theorem 3.12.** If a trans-para-Sasakian manifold is pseudo-projectively flat, then it is an Einstein manifold and a manifold of scalar curvature $\text{scal} = -2n(2n + 1)(\alpha^2 + \beta^2)$. 

Therefore

$$g(R(\xi, Y)\overline{C}(U, V)Z, \xi) - g(\overline{C}(R(\xi, Y)U, V)Z, \xi) - g(\overline{C}(U, R(\xi, Y)V)Z, \xi) - g(\overline{C}(U, V)R(\xi, Y)Z, \xi) = 0.$$
Proof. Suppose that \( \bar{P}(X, Y)Z = 0 \), then from (49), we get

\[
aR(X, Y)Z + b(Ric(Y, Z)X - Ric(X, Z)Y) - \frac{(a + 2nb)scal}{2n(2n + 1)}(g(Y, Z)X - g(X, Z)Y) = 0.
\]  

(50)

Taking the inner product on both sides of (50) by \( \xi \), we get

\[
-\frac{(a + 2nb)scal}{2n(2n + 1)}(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)) = 0.
\]

(51)

Putting \( X = \xi \) and using (10) and (17) in (51), we get

\[
-\alpha n(\alpha^2 + \beta^2)(g(Y, Z) - \eta(Y)\eta(Z)) + b(Ric(Y, Z) + 2n(\alpha^2 + \beta^2)\eta(Y)\eta(Z)) + \alpha n(2n + 1)(\alpha^2 + \beta^2) = 0.  
\]

(52)

From the identity (52), we obtain that \( Ric(X, Y) = -2n(\alpha^2 + \beta^2)g(Y, Z) \) and \( scal = -2n(2n + 1)(\alpha^2 + \beta^2) \).

\[\square\]

**Theorem 3.13.** A trans-para-Sasakian manifold is satisfying the relation \( R(X, Y)\bar{P} = 0 \) is an Einstein manifold and a manifold of scalar curvature \( scal = -2n(2n + 1)(\alpha^2 + \beta^2) \).

Proof. From equality (49), we have \( \eta(\bar{P}(X, Y)\xi) = 0 \). Now,

\[
(R(X, Y)\bar{P}(U, V))Z = R(X, Y)\bar{P}(U, V)Z - \bar{P}(R(X, Y)U, V)Z - \bar{P}(U, R(X, Y)V)Z - \bar{P}(U, V)R(X, Y)Z.
\]

By assumption \( R(X, Y)\bar{P} = 0 \), so we have

\[
R(X, Y)\bar{P}(U, V)Z - \bar{P}(R(X, Y)U, V)Z - \bar{P}(U, R(X, Y)V)Z - \bar{P}(U, V)R(X, Y)Z = 0.
\]

(53)

Therefore

\[
g(R(\xi, Y)\bar{P}(U, V)Z, \xi) - g(\bar{P}(R(\xi, Y)U, V)Z, \xi) - g(\bar{P}(U, R(\xi, Y)V)Z, \xi) - g(\bar{P}(U, V)R(\xi, Y)Z, \xi) = 0.
\]

From this, it follows that

\[
-\bar{P}(U, V, Z, Y) + \eta(\xi)\eta(\bar{P}(U, V)Z) - \eta(U)\eta(\bar{P}(Y, V)Z) + \eta(Y, U)\eta(\bar{P}(\xi, V)Z) - \eta(Y)\eta(\bar{P}(U, \xi)Z) + g(Y, V)\eta(\bar{P}(U, \xi)Z) - \eta(Z)\eta(\bar{P}(U, V)Y) = 0.
\]

(54)

Let \( \{\xi_i\} \), \( i = 1, ..., 2n + 1 \) be an orthonormal basis. Then summing up for \( 1 \leq i \leq 2n + 1 \) of the relation (54) for \( Y = U = \xi \) yields

\[
\bar{P}(\xi_i, V, Z, \xi) - 2n\eta(\bar{P}(\xi, V)Z) + \eta(Z)\eta(\bar{P}(\xi_i, V)\xi) = 0.
\]

(55)

Taking the trace of the identity, we obtain

\[
-\bar{P}(\xi_i, V, Z, \xi) + 2n\bar{P}(\xi, V, Z, \xi) + \eta(Z)\bar{P}(\xi, \xi, \xi, \xi) = 0.
\]

(56)

From identity (56), we get

\[
aRic(V, Z) = -2n(\alpha^2 + \beta^2)g(V, Z) + (b.scal + 2n(2n + 1)b(\alpha^2 + \beta^2))\eta(V)\eta(Z).
\]

(57)

Taking \( Z = \xi \) in (57) and using (17) we obtain

\[
scal = -2n(2n + 1)(\alpha^2 + \beta^2) \text{ and } Ric(V, Z) = -2n(\alpha^2 + \beta^2)g(V, Z).
\]

(58)

\[\square\]
The PC-Bochner curvature tensor on $M$ is defined by [9]

$$B(X, Y, Z, W) = R(X, Y, Z, W) + \frac{1}{2n + 4} (Ric(X, Z)g(Y, W) - Ric(Y, Z)g(X, W) + \text{Ric}(Y, W)g(X, Z) - \text{Ric}(X, W)g(Y, Z) + \text{Ric}(\varphi X, Z)g(Y, \varphi W) - \text{Ric}(\varphi Y, Z)g(X, \varphi W) - \text{Ric}(\varphi X, W)g(Y, \varphi Z) - \text{Ric}(\varphi Y, W)g(X, \varphi Z) + 2\text{Ric}(\varphi X, Y)g(Z, \varphi W) + 2\text{Ric}(\varphi Z, W)g(X, \varphi Y) - \text{Ric}(X, Z)\eta(Y)\eta(W) + \text{Ric}(Y, Z)\eta(X)\eta(W) - \text{Ric}(X, W)\eta(Y)\eta(Z) + \text{Ric}(\varphi Z, W)g(X, \varphi Y) - \text{Ric}(X, Z)\eta(Y)\eta(W) + \text{Ric}(Y, Z)\eta(X)\eta(W) - \text{Ric}(X, W)\eta(Y)\eta(Z)) + \frac{k - 4}{2n + 4} (g(X, Z)g(Y, W) - g(Y, Z)g(X, W)) - \frac{k + 2n}{2n + 4} (g(Y, \varphi W)g(X, \varphi Z) - g(X, \varphi W)g(Y, \varphi Z) + 2g(X, \varphi Y)g(Z, \varphi W)) - \frac{k}{2n + 4} (g(X, Z)\eta(Y)\eta(W) - g(Y, \varphi W)g(X, \varphi Z) - g(Y, W)\eta(X)\eta(Z) - g(X, W)\eta(Y)\eta(Z)),
$$

where $k = -\frac{\alpha^2 + \beta^2}{2n + 4}$.

Using the PC-Bochner curvature tensor we have

**Theorem 3.14.** If a trans-para-Sasakian manifold is paracontact conformally flat, then $\alpha^2 + \beta^2 = 1$.

**Proof.** Suppose that the manifold is paracontact conformally flat. Then the condition $B(X, Y)Z = 0$ holds. Putting $X = Z = \xi$ and using (11), we obtain

$$ (\alpha^2 + \beta^2 - 1)(Y - \eta(Y)\xi) = 0. \quad (59) $$

Since $Y - \eta(Y)\xi = \varphi^2 Y \neq 0$, we have $\alpha^2 + \beta^2 - 1 = 0$. □

**Theorem 3.15.** If a trans-para-Sasakian manifold satisfies the condition $B(\xi, Y)\text{Ric} = 0$, then it is either an Einstein manifold with scalar curvature scal = $-2n(2n + 1)(\alpha^2 + \beta^2)$ or $\alpha^2 + \beta^2 = 1$.

**Proof.** Suppose that the condition $B(\xi, Y)\text{Ric}(Z, V) = 0$ holds. This condition implies that

$$ \text{Ric}(B(\xi, Y)Z, V) + \text{Ric}(Z, B(\xi, Y)V) = 0. \quad (60) $$

Putting $V = \xi$ and using (11), we obtain

$$ (\alpha^2 + \beta^2 - 1)(\text{Ric}(Y, Z) + 2n(\alpha^2 + \beta^2)g(Y, Z)) = 0. \quad (61) $$

□

**References**


