Strong Convergent Iterative Techniques for 2-generalized Hybrid Mappings and Split Equilibrium Problems

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Abstract. In this paper, we suggest a new iterative scheme for finding a common element of the set of solutions of a split equilibrium problem and the set of fixed points of 2-generalized hybrid mappings in Hilbert spaces. We show that the iteration converges strongly to a common solution of the considered problems. A numerical example is illustrated to verify the validity of the proposed algorithm. The results obtained in this paper extend and improve some known results in the literature.

1. Introduction

Let $H_1$ and $H_2$ be real Hilbert spaces with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. $F_1: C_1 \times C_1 \to \mathbb{R}$ and $F_2: C_2 \times C_2 \to \mathbb{R}$ are two equilibrium functions, where $C_1$ and $C_2$ are nonempty closed convex subsets of $H_1$ and $H_2$, respectively. If $A: H_1 \to H_2$ is a bounded linear operator, then split equilibrium problem (SEP) is defined as follows:

Find $x^* \in C_1$ such that

$$F_1(x^*, x) \geq 0 \quad \forall x \in C_1, \tag{1}$$

and $y^* = Ax^* \in C_2$ such that

$$F_2(y^*, y) \geq 0 \quad \forall y \in C_2. \tag{2}$$

The set of all solutions of this split equilibrium problem is denoted by $\Omega$, i.e,

$$\Omega = \{ z \in C : z \in EP(F_1) \text{ such that } Az \in EP(F_2) \}.$$
where \( EP(F_1) \) and \( EP(F_2) \) denote the sets of all solutions of the equilibrium problems (1) and (2), respectively.

Equilibrium problem has received much attention due to its applications in a large variety of problems arising in physics, optimizations, economics and some others. The split equilibrium problem (1)-(2) constitute a pair of equilibrium problems where is the generalization of split feasibility problems. Some iterative methods have been rapidly established for solving these problems (see [1-10]).

Let \( H \) be a real Hilbert space and \( C \) be a nonempty closed convex subset of \( H \). A mapping \( T : C \to C \) is said to be:

1. nonexpansive if \( \|T(x) - T(y)\| \leq \|x - y\|, \forall x, y \in C \);
2. quasi-nonexpansive if \( \|T(x) - p\| \leq \|x - p\| \) for all \( x \in C \) and \( p \in F(T) \), where \( F(T) \) denotes the set of fixed points of \( T \);
3. nonspreading if
   \[
   2\|T(x) - T(y)\|^2 \leq \|T(x) - y\|^2 + \|T(y) - x\|^2, \forall x, y \in C;
   \]
4. firmly nonexpansive if
   \[
   \|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle, \forall x, y \in C;
   \]

It is obvious that the above inequality is equivalent to

\[
\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \forall x, y \in C;
\]

5. \( \alpha \)-inverse strongly monotone if there exists \( \alpha > 0 \) such that
   \[
   (x - y, Tx - Ty) \geq \alpha\|Tx - Ty\|^2, \forall x, y \in C;
   \]
6. hybrid if
   \[
   3\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \forall x, y \in C;
   \]
7. \((\alpha, \beta)\)-generalized hybrid if there exist \( \alpha, \beta \in \mathbb{R} \) such that
   \[
   \alpha\|T(x) - T(y)\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - Ty\|^2 + (1 - \beta)\|x - y\|^2, \forall x, y \in C;
   \]
8. 2-generalized hybrid mapping if there exist \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R} \) such that for all \( x, y \in C \)
   \[
   \alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \leq \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2,
   \]
such a mapping is called a \((\alpha_1, \alpha_2, \beta_1, \beta_2)\)-generalized hybrid mapping.

It is also easy to see that
a. \((1, 0)\)-generalized hybrid mapping is nonexpansive.
B. \((2, 1)\)-generalized hybrid mapping is nonspreading.
C. \((3, 2, 1/2)\)-generalized hybrid mapping is hybrid.
D. \((0, \alpha_2, 0, \beta_2)\)-generalized hybrid mapping is \((\alpha_2, \beta_2)\)-generalized hybrid.
E. 2-generalized hybrid mapping is quasi-nonexpansive.

In [11], Hojo et al. give two examples of 2-generalized hybrid mappings which are not generalized hybrid mapping.

Recently, the existence of fixed points and the convergence theorems of hybrid mappings have been studied by many authors (see [12-20]).

Very recently, Alizadeh and Moradlou [21-23] have obtained some weak convergence theorems for 2-generalized hybrid mapping and equilibrium problems.

Motivated by the above works, in this paper we introduce and consider a new iterative algorithm for a common element of the sets of solutions of the split equilibrium problems and common fixed points of 2-generalized hybrid mapping in Hilbert spaces. Under suitable conditions, some strong convergence for the sequences generated by the algorithm to a common solution of the problems is proved. The results presented in the paper extend and improve the corresponding results announced by Alizadeh and Moradlou [21], and some others.
2. Preliminaries and lemmas

In this section, we give some definitions and preliminaries which will be used in the sequel.

Let $H$ be a real Hilbert space and $C$ be a nonempty closed convex subset of $H$. The operator $P_C$ denotes the metric projection from $H$ onto $C$. It is a fact that $P_C$ is a firmly nonexpansive mapping from $H$ onto $C$. Further, for any $x \in H$, $z = P_Cx$ if and only if $(x - z, z - y) \geq 0$, $\forall y \in C$.

**Lemma 2.1 ([24]).** Let $H$ be a real Hilbert space and $T: H \to H$ be a nonexpansive mapping. Then for all $(x, y) \in H \times F(T)$, we have

$$\langle x - T(x), y - T(x) \rangle \leq \frac{1}{2}\|x - T(x)\|^2.$$

**Lemma 2.2 (Demiclosedness principle).** Let $T$ be a nonexpansive mapping on a closed convex subset $C$ of a real Hilbert space $H$. Then $I - T$ is demiclosed at any point $y \in H$, that is, if $x_n \to x$ and $x_n - Tx_n \rightharpoonup y \in H$, then $x - Tx = y$.

To obtain our main results, we need the following assumptions.

**Assumption 2.3 ([25, 26]).** Let $F: C \times C \to \mathbb{R}$ be an equilibrium function satisfying the following assumptions:

1. $F(x, x) = 0$, $\forall x \in C$;
2. $F$ is monotone, i.e., $F(x, y) + F(y, x) \leq 0$, $\forall x, y \in C$;
3. $F$ is hemicontinuous with respect to the first variable, i.e., for each $x, y, z \in C$, $\limsup F(tz + (1-t)x, y) \leq F(x, y)$;
4. for each $x \in C$, the function $y \mapsto F(x, y)$ is convex and lower semi-continuous.

**Lemma 2.4 ([27]).** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $F: C \times C \to \mathbb{R}$ be an equilibrium function which satisfies the Assumption 2.3. Then for all $r > 0$, the resolvent of the equilibrium function $T^F_r : H \to C$ defined by

$$T^F_r(x) = \{z \in C : F(z, y) + \frac{1}{r}(y - z, z - x) \geq 0, \forall y \in C\}, \forall x \in H,$$

is well defined and satisfies the following conditions:

1. $T^F_r(x)$ is nonempty and single-valued for each $x \in H$;
2. $T^F_r$ is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T^F_r(x) - T^F_r(y)\|^2 \leq \langle T^F_r(x) - T^F_r(y), x - y \rangle;$$
3. $F(T^F_r) = EP(F)$;
4. the set $EP(F)$ is closed and convex;
5. for $r, s > 0$ and for all $x, y \in H$, one has

$$\|T^F_r(x) - T^F_s(y)\|^2 \leq \|x - y\|^2 + |1 - \frac{r}{s}|\|T^F_r(x) - x\|^2.$$

**Lemma 2.5 ([28]).** Let $H$ be a a real Hilbert space. For all $x, y \in H$, $\|ax + (1 - a)y\|^2 = a\|x\|^2 + (1 - a)\|y\|^2 - a(1 - a)\|x - y\|^2$, $\forall a \in \mathbb{R}$.

Now, we give a new iterative scheme as follows:

Let $C_1$ be a nonempty closed convex subset of a real Hilbert space $H_1$ and $S : C_1 \to C_1$ is a 2-generalized hybrid mapping.

For an initial point $x_0 \in C_1$, let $x_1 = P_{C_1}x_0$ and $D_1 = C_1$. Then

$$\begin{align*}
\alpha_n &= T^F_r[I - \gamma A^*(I - T^F_r)A]x_n, \\
\beta_n &= (1 - \frac{\beta_n}{\alpha_n})u_n + \frac{\beta_n}{\alpha_n} \sum_{k=0}^{n-1} S^k u_n, \\
y_n &= (1 - \alpha_n)u_n + \frac{\alpha_n}{\beta_n} \sum_{k=0}^{n-1} S^k v_n, \\
D_{n+1} &= \{x \in D_n : \|y_n - x\| \leq \|x_n - x\|\}, \\
x_{n+1} &= P_{D_{n+1}}x_0, \forall n \geq 1.
\end{align*}$$

(4)

For this iterative scheme, we will discuss its strong convergence and also prove that its limit point belongs to $F(S) \cap \Omega$, where $F(S)$ is a set of fixed points of $S$. 

3. The main results

In this section, we show some strong convergence theorems for finding a common element of the solution set of split equilibrium problems and the set of fixed points of 2-generalized hybrid mapping in a Hilbert space.

Throughout this section we need the following assumptions:
(A1) $C_1 \subset H_1$ and $C_2 \subset H_2$ are nonempty closed convex subsets of the real Hilbert spaces $H_1$ and $H_2$, respectively.
(A2) $A : H_1 \to H_2$ is a bounded linear mapping.
(A3) $S : C_1 \to C_1$ is a 2-generalized hybrid mapping.
(A4) $F_1 : C_1 \times C_1 \to R$, $F_2 : C_2 \times C_2 \to R$ are two equilibrium functions such that Assumption 2.3 holds.
(A5) $T_n : H_1 \to C_1$, $T_n^2 : H_2 \to C_2$ are the resolvent of the equilibrium functions $F_1$ and $F_2$, respectively.
We also need the following lemma.

**Lemma 3.1 ([26]).** Assume that the assumptions (A1–A5) are satisfied and $r_n \in (r, +\infty)$ with $r > 0$, $\gamma \in (0, \frac{1}{L})$, where $L$ is the spectral radius of $A^*A$. Then $A^*(I - T_n^2)A$ is a $\frac{1}{2}$-inverse strongly monotone mapping and $I - \gamma A^*(I - T_n^2)A$ is a nonexpansive mapping.

**Theorem 3.2.** Assume that the assumptions (A1 – A5) are satisfied and $0 < \alpha < \alpha_1, \beta_1 < \beta < 1$, $r > r_n < \infty$, for $a, \beta \in (0, 1)$, $r > 0$, $\gamma \in (0, \frac{1}{L})$, where $L$ is the spectral radius of $A^*A$. In addition, if $\Theta = F(S) \cap \Omega \neq \emptyset$, then for any $x_0 \in C_1$, the sequence $\{x_n\}$ defined by (4) converges strongly to some point $p \in \Theta$.

**Proof.** We shall divide the proof into five steps.

**Step (1):** $\Theta \subset D_n$, $\forall n \geq 1$.

Obviously, $\Theta \subset D_1 = C_1$. By induction, assume that $\Theta \subset D_n$ for some $n \geq 1$. We only need to show that $\Theta \subset D_{n+1}$. For any $p \in \Theta$, we have $p = T_n^2 p$ and $(I - \gamma A^*(I - T_n^2)A)p = p$ from Lemma 2.4. The Lemma 3.1 results in

$$
\|u_n - p\| = \|T_n^2(I - \gamma A^*(I - T_n^2)A)x_n - T_n^2(I - \gamma A^*(I - T_n^2)A)p\|
\leq \|(I - \gamma A^*(I - T_n^2)A)x_n - (I - \gamma A^*(I - T_n^2)A)p\|
\leq \|x_n - p\|.
$$

(5)

Since $p \in F(S)$ and $S$ is quasi-nonexpansive, we get

$$
\|Sv_n - p\| \leq \|v_n - p\|
= \|(1 - \beta_n)u_n + \frac{\beta_n}{n} \sum_{k=0}^{n-1} S^k u_n - p\|
\leq (1 - \beta_n)\|u_n - p\| + \frac{\beta_n}{n} \sum_{k=0}^{n-1} \|S^k u_n - p\|
\leq (1 - \beta_n)\|u_n - p\| + \frac{\beta_n}{n} \sum_{k=0}^{n-1} \|S u_n - p\|
\leq (1 - \beta_n)\|u_n - p\| + \frac{\beta_n}{n} \sum_{k=0}^{n-1} \|S u_n - p\|
\leq \|u_n - p\|.
$$

(6)
Combining (4), (5), (6) and Lemma 2.5, we obtain

\[
\|y_n - p\|^2 = \|(1 - \alpha_n)u_n + \frac{\alpha_n}{n} \sum_{k=0}^{n-1} s^k v_n - p\|^2
\]

\[
= \|(1 - \alpha_n)(u_n - p) + \alpha_n\left(\frac{1}{n} \sum_{k=0}^{n-1} s^k v_n - p\right)\|^2
\]

\[
= (1 - \alpha_n)\|u_n - p\|^2 + \alpha_n\left[\frac{1}{n} \sum_{k=0}^{n-1} (s^k v_n - p)\right]^2 - \alpha_n(1 - \alpha_n)\|u_n - s\|_2^2
\]

\[
\leq (1 - \alpha_n)\|u_n - p\|^2 + \alpha_n\|v_n - p\|^2 - \alpha_n(1 - \alpha_n)\|u_n - s\|_2^2
\]

\[
\leq (1 - \alpha_n)\|u_n - p\|^2 + \alpha_n\|v_n - p\|^2 - \alpha_n(1 - \alpha_n)\|u_n - \frac{1}{n} \sum_{k=0}^{n-1} s^k v_n\|^2
\]

\[
\leq \|u_n - p\|^2 - \alpha_n(1 - \alpha_n)\|u_n - \frac{1}{n} \sum_{k=0}^{n-1} s^k v_n\|^2
\]

\[
\leq \|x_n - p\|^2,
\]

which implies that \(p \in D_{n+1}\). Therefore, \(\Theta \subset D_{n+1}\).

**Step (II):** The sequence \(\{x_n\}\) is a Cauchy sequence.

According to (4) and the Step (I), it is obvious that \(D_n\) is nonempty closed and convex subset of \(C_1\). Since \(\Theta \subset D_{n+1} \subset D_n\), for all \(n \geq 1\), we obtain from \(x_{n+1} = P_{D_{n+1}}x_0\) that

\[
\|x_{n+1} - x_0\| = \|P_{D_{n+1}}x_0 - x_0\| \leq \|p - x_0\|, \forall p \in \Theta,
\]

which implies that \(\{x_n\}\) is bounded. By \(x_n = P_{D_{n+1}}x_0\), we have

\[
\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0.
\]

Therefore,

\[
0 \leq \langle x_0 - x_n, x_n - x_{n+1} \rangle \leq -\|x_n - x_0\|^2 + \|x_{n+1} - x_0\||x_n - x_n||x_0 - x_n||.
\]

Hence,

\[
\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \forall n \geq 1,
\]

which implies that \(\lim_{n \to \infty} \|x_n - x_0\|\) exists. For any \(n > m \geq 1\), \(x_m = P_{D_m}x_0\), we also have

\[
\|x_n - x_0\|^2 = \|x_n - x_m + x_m - x_0\|^2
\]

\[
= \|x_n - x_m\|^2 + \|x_m - x_0\|^2 + 2\langle x_n - x_m, x_m - x_0 \rangle
\]

\[
\geq \|x_n - x_m\|^2 + \|x_m - x_0\|^2.
\]

Therefore, we get

\[
\|x_n - x_m\|^2 \leq \|x_n - x_0\|^2 - \|x_m - x_0\|^2 \to 0 \quad \text{as } n, m \to \infty.
\]

Hence \(\{x_n\}\) is a Cauchy sequence. We may assume that

\[
x_n \to x', \quad \text{as } n \to \infty.
\]

**Step (III):** \(\lim_{n \to \infty} \|u_n - \frac{1}{n} \sum_{k=0}^{n-1} s^k u_n\| = 0\).
Since \( x_{n+1} \in D_{n+1} \subseteq D_n \), by the definition of \( D_{n+1} \), we have

\[
\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|.
\]

It follows from (8) that

\[
\lim_{n \to \infty} \|y_n - x_{n+1}\| = \lim_{n \to \infty} \|x_n - x_{n+1}\| = 0. \tag{10}
\]

Therefore, we obtain from (8) and (10)

\[
\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{11}
\]

Further, from (7), we have

\[
\|y_n - p\|^2 \leq \|u_n - p\|^2 - \alpha_n(1 - \alpha_n)\|u_n - \frac{1}{n} \sum_{k=0}^{n-1} S^k v_n\|^2. \tag{12}
\]

Therefore,

\[
\alpha_n(1 - \alpha_n)\|u_n - \frac{1}{n} \sum_{k=0}^{n-1} S^k v_n\|^2 \leq \|u_n - p\|^2 - \|y_n - p\|^2,
\]

\[
\leq \|x_n - p\|^2 - \|y_n - p\|^2,
\]

\[
\leq \|x_n - y_n\| \|(x_n - p) + (y_n - p)\|.
\]

By \( 0 < \alpha < \alpha_n < \beta < 1 \) and (11), we have

\[
\|u_n - \frac{1}{n} \sum_{k=0}^{n-1} S^k v_n\| \to 0, \quad \text{as} \quad n \to \infty. \tag{13}
\]

Furthermore, \( p \in \Theta \) ensures \( p = T_{r_n}^p p \) and \( p = (I - \gamma A'(I - T_{r_n}^{F_p})A)p \). Therefore, we have from Lemma 3.1 and (3)

\[
\|u_n - p\|^2 = \|T_{r_n}^{F_p}[(I - \gamma A'(I - T_{r_n}^{F_p})A)x_n - T_{r_n}^{F_p}[(I - \gamma A'(I - T_{r_n}^{F_p})A)p]\|^2
\]

\[
\leq \|[(I - \gamma A'(I - T_{r_n}^{F_p})A)x_n - [(I - \gamma A'(I - T_{r_n}^{F_p})A)p]^2
\]

\[
-[(I - T_{r_n}^{F_p})[(I - \gamma A'(I - T_{r_n}^{F_p})A)x_n - (I - T_{r_n}^{F_p})[(I - \gamma A'(I - T_{r_n}^{F_p})A)p]^2
\]

\[
= \|x_n - p\|^2 + \gamma^2 \|x_n - p, A'(I - T_{r_n}^{F_p})Ax_n - A'(I - T_{r_n}^{F_p})Ap\|^2
\]

\[
+ \gamma^2 \|A'(I - T_{r_n}^{F_p})Ax_n - A'(I - T_{r_n}^{F_p})Ap\|^2
\]

\[
-[(I - T_{r_n}^{F_p})[(I - \gamma A'(I - T_{r_n}^{F_p})A)x_n - (I - T_{r_n}^{F_p})[(I - \gamma A'(I - T_{r_n}^{F_p})A)p]^2
\]

\[
\leq \|x_n - p\|^2 + \gamma^2 \|x_n - p, A'(I - T_{r_n}^{F_p})Ax_n - A'(I - T_{r_n}^{F_p})Ap\|^2 - [(I - T_{r_n}^{F_p})[(I - \gamma A'(I - T_{r_n}^{F_p})A)x_n\|^2. \tag{14}
\]

From (12) and (14), we get

\[
\gamma \left( \frac{2}{L} - \gamma \right) \|A'(I - T_{r_n}^{F_p})Ax_n\|^2 + \|[(I - T_{r_n}^{F_p})[(I - \gamma A'(I - T_{r_n}^{F_p})A)x_n\|^2
\]

\[
\leq \|x_n - p\|^2 + \|u_n - p\|^2
\]

\[
\leq \|x_n - y_n\| \|(x_n - p) + (y_n - p)\|.
\]

By \( \gamma \in (0, \frac{1}{L}) \), using (11), we have

\[
\lim_{n \to \infty} \|A'(I - T_{r_n}^{F_p})Ax_n\| = 0, \quad \lim_{n \to \infty} \|[(I - T_{r_n}^{F_p})[(I - \gamma A'(I - T_{r_n}^{F_p})A)x_n\| = 0. \tag{16}
\]
Since $T_{r_k}^f$ is firmly nonexpansive, so we have
\[
\|u_n - p\|^2 = \|T_{r_k}^f[I - \gamma A^*(I - T_{r_k}^f)A]x_n - T_{r_k}^f p\|^2 \\
\leq \|I - \gamma A^*(I - T_{r_k}^f)A\|\|x_n - p\|^2 \\
= \|x_n - p\|^2 + \gamma^2 \|A^*(I - T_{r_k}^f)A\|\|x_n - x_n^*\|^2 + 2\gamma \langle p - x_n, A^*(I - T_{r_k}^f)Ax_n \rangle \\
= \|x_n - p\|^2 + \gamma^2 \|A^*(I - T_{r_k}^f)A\|\|x_n - x_n^*\|^2 + 2\gamma \langle p - Ax_n, (I - T_{r_k}^f)Ax_n \rangle,
\]
and
\[
\gamma^2 \|A^*(I - T_{r_k}^f)Ax_n\|^2 = \gamma^2 (\|I - T_{r_k}^f)Ax_n, AA^*(I - T_{r_k}^f)Ax_n) \\
\leq L\gamma^2 \|I - T_{r_k}^f)Ax_n\|^2.
\]
We also have from Lemma 2.1
\[
2\gamma \langle p - Ax_n, (I - T_{r_k}^f)Ax_n \rangle = 2\gamma \langle p - T_{r_k}^fAx_n - (Ax_n - T_{r_k}^fAx_n), (I - T_{r_k}^f)Ax_n \rangle \\
= 2\gamma (\|p - T_{r_k}^fAx_n\|^2 - \|Ax_n - T_{r_k}^fAx_n\|^2) \\
\leq 2\gamma \|p\|^2 - \|Ax_n - T_{r_k}^fAx_n\|^2,
\]
which implies from (12) that
\[
-\gamma (L\gamma - 1)\|I - T_{r_k}^f)Ax_n\|^2 \leq \|x_n - p\|^2 - \|u_n - p\|^2 \\
\leq \|x_n - p\|^2 - \|y_n - p\|^2 \\
\leq \|x_n - y_n\| (\|x_n - p\| + \|y_n - p\|).
\]
Due to $\gamma \in (0, \frac{1}{L})$ and (11), we have
\[
\lim_{n \to \infty} \|I - T_{r_k}^f)Ax_n\| = 0. \tag{17}
\]
Hence, we obtain from (16)
\[
\|u_n - x_n\| = \|T_{r_k}^f[I - \gamma A^*(I - T_{r_k}^f)A]x_n - x_n\| \\
\leq \|T_{r_k}^f[I - \gamma A^*(I - T_{r_k}^f)A]x_n - I - \gamma A^*(I - T_{r_k}^f)A]x_n\| + \|I - \gamma A^*(I - T_{r_k}^f)A]x_n - x_n\| \\
= \|I - T_{r_k}^f\| \|I - \gamma A^*(I - T_{r_k}^f)A]x_n\| + \|A^*(I - T_{r_k}^f)A]x_n\| \to 0, \quad \text{as } n \to \infty. \tag{18}
\]
Using (4) and Lemma 2.5, we have
\[
\|v_n - p\|^2 = \|(1 - \beta_n)u_n + \frac{\beta_n}{n} \sum_{k=0}^{n-1} S^k u_n - p\|^2 \\
= \|(1 - \beta_n)(u_n - p) + \beta_n (\frac{1}{n} \sum_{k=0}^{n-1} S^k u_n - p)\|^2 \\
= (1 - \beta_n)\|u_n - p\|^2 + \beta_n \|\frac{1}{n} \sum_{k=0}^{n-1} (S^k u_n - p)\|^2 - \beta_n (1 - \beta_n)\|u_n - \frac{1}{n} \sum_{k=0}^{n-1} S^k u_n\|^2
\]
\[
\|y_n - p\|^2 = \|(1 - \alpha_n)y_n + \frac{\alpha_n}{n} \sum_{k=0}^{n-1} S^k v_n - p\|^2
\]

\[
= \|(1 - \alpha_n)(u_n - p) + \frac{\alpha_n}{n} \sum_{k=0}^{n-1} S^k v_n - p\|^2
\]

\[
= (1 - \alpha_n)\|u_n - p\|^2 + \frac{\alpha_n}{n} \sum_{k=0}^{n-1} \|S^k v_n - p\|^2 - \alpha_n(1 - \alpha_n)\|u_n - \frac{1}{n} \sum_{k=0}^{n-1} S^k v_n\|^2
\]

\[
\leq (1 - \alpha_n)\|u_n - p\|^2 + \frac{\alpha_n}{n} \sum_{k=0}^{n-1} \|S^k v_n - p\|^2 - \alpha_n(1 - \alpha_n)\|u_n - \frac{1}{n} \sum_{k=0}^{n-1} S^k v_n\|^2
\]

\[
\leq (1 - \alpha_n)\|u_n - p\|^2 + \alpha_n\|v_n - p\|^2 - \alpha_n(1 - \alpha_n)\|u_n - \frac{1}{n} \sum_{k=0}^{n-1} S^k v_n\|^2
\]

\[
\leq (1 - \alpha_n)\|u_n - p\|^2 + \alpha_n\|v_n - p\|^2 - \alpha_n(1 - \alpha_n)\|u_n - \frac{1}{n} \sum_{k=0}^{n-1} S^k v_n\|^2
\]

\[
\leq (1 - \alpha_n)\|u_n - p\|^2 - \alpha_n(1 - \beta_n)\|u_n - \frac{1}{n} \sum_{k=0}^{n-1} S^k u_n\|^2
\]

\[
\leq \|x_n - p\|^2 - \alpha_n(1 - \beta_n)\|x_n - \frac{1}{n} \sum_{k=0}^{n-1} S^k u_n\|^2,
\]

(20)

which implies from (20) and \(0 < \alpha < \alpha_n < \beta < 1\) that

\[
\alpha_n(1 - \beta_n)\|u_n - \frac{1}{n} \sum_{k=0}^{n-1} S^k u_n\|^2 \leq \alpha_n(1 - \beta_n)\|u_n - \frac{1}{n} \sum_{k=0}^{n-1} S^k u_n\|^2
\]

\[
\leq \|x_n - p\|^2 - \|y_n - p\|^2
\]

\[
\leq (\|x_n - p\|^2 + \|y_n - p\|)(\|x_n - y_n\|).
\]

(21)

In virtue of \(0 < \alpha < \beta_n < \beta < 1\), (9), (11) and (21), we get

\[
\lim_{n \to \infty} \|u_n - \frac{1}{n} \sum_{k=0}^{n-1} S^k u_n\| = 0.
\]

(22)

**Step (IV):** \(x^* \in \Theta = F(S) \cap \Omega\), where \(x^*\) is the limit in (3.5).

To do so, we firstly show that \(x^* \in \Omega\).
By the boundedness of $A$ and (9), we get $Ax_n \to Ax$. Then we have from Lemma 2.4 and (17)

$$
\|T_{r_n}^F Ax_n - T_{r_n}^F Ax\| \leq \left| 1 - \frac{r}{r_n} \right| \|T_{r_n}^F Ax_n - Ax_n\| \to 0, \quad \text{as } n \to \infty
$$

and

$$
\|T_{r_n}^F Ax_n - Ax_n\| \leq \|T_{r_n}^F Ax_n - T_{r_n}^F Ax\| + \|T_{r_n}^F Ax - Ax\| \to 0, \quad \text{as } n \to \infty.
$$

Since $T_{r_n}^F$ is nonexpansive, we easily get from Lemma 2.2 and Lemma 2.4

$$
T_{r_n}^F Ax' = Ax', \quad \text{i.e., } Ax' \in F(T_{r_n}^F) = EP(F_2).
$$

Let $w_n = (I - \gamma A'(I - T_{r_n}^F)A)x_n$. By (16) we have

$$
\|w_n - x_n\| = \|y A'(I - T_{r_n}^F)Ax_n\| \to 0, \quad \text{as } n \to \infty.
$$

We also have from (16)

$$
\|T_{r_n}^F w_n - T_{r_n}^F w\| \leq \left| 1 - \frac{r}{r_n} \right| \|T_{r_n}^F w_n - w_n\| \to 0, \quad \text{as } n \to \infty.
$$

Therefore,

$$
\|T_{r_n}^F w_n - w\| \leq \|T_{r_n}^F w_n - T_{r_n}^F w\| + \|T_{r_n}^F w_n - w_n\| \to 0, \quad \text{as } n \to \infty.
$$

Since $T_{r_n}^F$ is nonexpansive, we get from Lemma 2.2 and Lemma 2.4

$$
T_{r_n}^F x' = x', \quad \text{i.e., } x' \in F(T_{r_n}^F) = EP(F_1).
$$

Therefore, $x' \in \Omega$.

Now, we prove that $x' \in F(S)$.

By means of (18) and (22), we easily get from $x_n \to x'$

$$
\frac{1}{n} \sum_{k=0}^{n-1} S^k u_n \to x', \quad \text{as } n \to \infty.
$$

(23)

Since $S$ is a 2-generalized hybrid mapping, there exist $a_1, a_2, b_1, b_2 \in R$ such that for all $x, y \in C_1$

$$
a_1\|S^2x - y\|^2 + a_2\|Sx -Sy\|^2 \geq \left( a_1 - a_2 \right)\|x -Sy\|^2
$$

$$
+ b_2\|Sx - y\|^2 + (1 - a_1 - a_2)\|x - y\|^2.
$$

Since $F(S) \neq \emptyset$, then $S$ is quasi-nonexpansive. So $\|S^ku_n - p\| \leq \|u_n - p\| \leq \|x_n - p\|$, which implies that $\{S^ku_n\}$ is bounded. Since $S$ is a 2-generalized hybrid mapping, we have for all $y \in C_1$ and $k = 0, 2, 3, ..., n - 1$

$$
0 \leq b_1\|S^{k+2}x_n - y\|^2 + b_2\|S^{k+1}x_n - y\|^2 + (1 - b_1 - b_2)\|S^kx_n - y\|^2
$$

$$
- a_1\|S^{k+2}x_n - Sy\|^2 - a_2\|S^{k+1}x_n - Sy\|^2 - (1 - a_1 - a_2)\|S^kx_n - Sy\|^2
$$

$$
= b_1\|S^{k+2}x_n - Sy\|^2 + 2\|S^{k+2}x_n - Sy, Sy - y\| + \|Sy - y\|^2 + b_2\|S^{k+1}x_n - Sy\|^2
$$

$$
+ 2\|S^{k+1}x_n - Sy, Sy - y\| + \|Sy - y\|^2 + (1 - b_1 - b_2)\|S^kx_n - Sy\|^2
$$

$$
+ 2\|S^kx_n - Sy, Sy - y\| + \|Sy - y\|^2 - a_1\|S^{k+2}x_n - Sy\|^2 - a_2\|S^{k+1}x_n - Sy\|^2
$$

$$
- (1 - a_1 - a_2)\|S^kx_n - Sy\|^2
$$

$$
= \|Sy - y\|^2 + 2\|S^{k+2}x_n + b_2\|S^{k+1}x_n + (1 - b_1 - b_2)S^kx_n - Sy, Sy - y\|
$$

$$
+ (b_1 - a_1)\|S^{k+2}x_n - Sy\|^2 - \|S^kx_n - Sy\|^2 + (b_2 - a_2)\|S^{k+1}x_n - Sy\|^2 - \|S^kx_n - Sy\|^2.
$$

$$
\|Sy - y\|^2 + 2\|S^{k+2}x_n - Sy, Sy - y\| + 2\|S^{k+2}x_n - S^kx_n + b_2\|S^{k+1}x_n - S^kx_n, Sy - y\|
$$

$$
+ (b_1 - a_1)\|S^{k+2}x_n - Sy\|^2 - \|S^kx_n - Sy\|^2 + (b_2 - a_2)\|S^{k+1}x_n - Sy\|^2 - \|S^kx_n - Sy\|^2.
$$
Summing these inequalities from \( k = 0, 1, \ldots, n - 1 \) and diving by \( n \), we have by denoting \( z_n = \frac{1}{n} \sum_{k=0}^{n-1} S^k u_n \), for all \( n \geq 1 \)

\[
0 \leq \|S y - y\|^2 + 2(z_n - S y, S y - y) + 2 \frac{1}{n}(\beta_1(S^{n+1} x_n - S^n x_n - S x_n - x_n) + \beta_2(S^n x_n - x_n), S y - y)
\]

\[
+ (\beta_1 - \alpha_1) \frac{1}{n} \left( ||S^{n+1} x_n - S y||^2 + ||S^n x_n - S y||^2 - ||S x_n - S y||^2 - ||x_n - S y||^2 \right)
\]

\[
+ (\beta_2 - \alpha_2) \frac{1}{n} \left( ||S^n x_n - S y||^2 - ||x_n - S y||^2 \right).
\]

From (23) and the boundedness of \( \{S^u u_n\} \), we have

\[
0 \leq \|S y - y\|^2 + 2(x^* - S y, S y - y).
\]

Denote \( y = x^* \), we have

\[
0 \leq \|S x^* - x^*\|^2 + 2(x^* - S x^*, S x^* - x^*) = -\|S x^* - x^*\|^2.
\]

Hence \( x^* \in F(S) \). This shows that \( x^* \in \Theta \). □

4. Numerical Example

In this section, a numerical example will be illustrated to verify the validity of the proposed algorithm in Section 3.

**Example 4.1.** Consider the following split equilibrium problem driven by 2-generalized hybrid mapping \( S \): find \( x \in \mathbb{R} \) such that

\[
\begin{align*}
F_1(x, \bar{x}) & \geq 0, \forall \bar{x} \in C_1, \\
y & = A x \in C_2, \\
F_2(y, \bar{y}) & \geq 0, \forall \bar{y} \in C_2, \\
x & \in F(S),
\end{align*}
\]

(24)

where

\[
\begin{align*}
H_1 = H_2 &= \mathbb{R}, \\
C_1 &= [-3, 0], \\
C_2 &= [0, +\infty), \\
F_1(u, v) &= (u - 1)(v - u), \forall u, v \in C_1, \\
F_2(x, y) &= (x + 15)(y - x), \forall x, y \in C_2, \\
A x &= 3 x, \forall x \in \mathbb{R}, \\
S x &= \frac{1}{3} x, \forall x \in C_1.
\end{align*}
\]

By choosing

\[
\begin{align*}
\alpha_n &= \frac{1}{2} + \frac{1}{3n}, \\
\beta_n &= \frac{1}{3} - \frac{1}{6n}, \\
r &= 4, \\
\gamma &= \frac{1}{9}.
\end{align*}
\]

It is easy to check that \( F_1 \) and \( F_2 \) satisfy all conditions in Lemma 3.1, i.e. Assumption 2.3. Analogously to the Theorem 3.2, we abide by the following processes to obtain the solution of (24).
\[
\begin{align*}
  u_n &= \frac{1}{25} x_n, \\
  v_n &= \left(\frac{2}{3} + \frac{1}{6n}\right) u_n + \left(\frac{1}{n} + \frac{1}{-6n}\right) \sum_{k=0}^{n-1} \frac{1}{3^k} u_n, \\
  y_n &= \left(\frac{1}{2} - \frac{1}{3n}\right) u_n + \left(\frac{1}{n} + \frac{1}{2} + \frac{1}{-3n}\right) \sum_{k=0}^{n-1} \frac{1}{3^k} v_n.
\end{align*}
\]

It is easy to get that \(0 \in F(0) \cap \Omega\).
Moreover, numerical results in Table 1 for \(\{x_n\}\) also is demonstrated as follows.

**Table 1:**

<table>
<thead>
<tr>
<th>n</th>
<th>(x_0 = -2)</th>
<th>(n)</th>
<th>(x_0 = -1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-2.0000</td>
<td>1</td>
<td>1.0000</td>
</tr>
<tr>
<td>2</td>
<td>-2.0400</td>
<td>2</td>
<td>-5.2000 \times 10^{-1}</td>
</tr>
<tr>
<td>3</td>
<td>-5.3541</td>
<td>3</td>
<td>-2.6770 \times 10^{-1}</td>
</tr>
<tr>
<td>4</td>
<td>-2.7456 \times 10^{-1}</td>
<td>4</td>
<td>-1.3728 \times 10^{-1}</td>
</tr>
<tr>
<td>5</td>
<td>-1.4054 \times 10^{-1}</td>
<td>5</td>
<td>-7.0270 \times 10^{-2}</td>
</tr>
<tr>
<td>6</td>
<td>-7.1863 \times 10^{-2}</td>
<td>6</td>
<td>-3.5932 \times 10^{-2}</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>98</td>
<td>-9.1955 \times 10^{-29}</td>
<td>98</td>
<td>-4.5978 \times 10^{-29}</td>
</tr>
<tr>
<td>99</td>
<td>-4.6901 \times 10^{-29}</td>
<td>99</td>
<td>-2.3450 \times 10^{-29}</td>
</tr>
<tr>
<td>100</td>
<td>-2.3921 \times 10^{-29}</td>
<td>100</td>
<td>-1.1960 \times 10^{-29}</td>
</tr>
</tbody>
</table>

See table 1 for the values \(x_0 = -2\) or \(x_0 = -1\), we obtain \(x_n \to 0\), as \(n \to \infty\).

See Figure 1 and Figure 2 for the values \(x_0 = x_1 = -1\) and \(x_0 = x_1 = -2\). The computations associated with example were performed using MATLAB software.

![Figure 1](image1.png)

**Figure 1:** A plot of \(x_n, n = 0, 1, 2, \cdots, 100\), for Example 4.1
References


