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A SYNTHESIS OF THE UZAWA-LUCAS MODEL WITH THE WALRASIAN-GENERAL-EQUILIBRIUM AND NEOCLASSICAL-GROWTH THEORIES

ABSTRACT: This paper proposes a dynamic economic model of wealth accumulation and human capital accumulation with endogenous education. It is an extension of the Uzawa-Lucas model of a heterogeneous household economy with multiple ways of human capital accumulation. In addition to learning by education in the Uzawa-Lucas model (Uzawa, 1965; Lucas, 1988), we also consider Arrow’s ‘learning by producing’ (Arrow, 1962) and Zhang’s ‘learning by consuming’ (creative learning, Zhang, 2007) in the human capital accumulation equation. The economic system consists of one production sector and one education sector. Households differ in propensity to save, to obtain education, to consume, and in learning abilities. The model describes a dynamic interdependence between wealth accumulation, human capital accumulation, and division of labour with endogenous wealth and income distribution in perfect competition. We simulate the model to demonstrate the existence of equilibrium points and the motion of the dynamic system. We also demonstrate how changes in the propensity to obtain education, the population, the propensity to save, and the education sector’s total productivity affect economic development.

KEY WORDS: learning by producing; learning by consuming; learning by education; wealth and income distribution; heterogeneous households

JEL CLASSIFICATION: O41

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1. INTRODUCTION

Dynamic interdependence between economic growth and human capital is currently a leading topic in economic theory and empirical research. It has become evident that it is not enough to be only concerned with capital accumulation as in neoclassical growth theory in order to explain why countries grow differently. As observed by Easterlin (1981), in 1850 there were few people outside North-Western Europe and North America who had any formal education. The spread of formal school seems to have preceded the beginning of modern economic growth. In modern economies, human capital is a key determinant of economic growth (Hanushek and Kimko, 2000; Barro, 2001; Krueger and Lindahl, 2001; Castelló-Climent and Hidalgo-Cabrillana, 2012; and Barro and Lee, 2013). There are many studies on the dynamic interdependence between education and economic growth. Mincer (1974) published the seminal work on estimating the impact of education on earnings in 1974. He finds that for white males not working on farms, an extra year of education raises the earnings by about 7%. Earlier studies (e.g., Tilak, 1989) show that the spread of education can substantially reduce inequality within countries. Could et al. (2001) build a model to provide insights into the evolution of wage inequality within and between industries and education groups in recent decades. The model shows that increasing randomness is the primary source of inequality growth among uneducated workers, but inequality growth among educated workers is determined more by changes in composition and return to ability (which is closely related to education). Tselios (2008) studies the relationship between income and educational inequalities in the regions of the European Union, using the European Community Household Panel data survey for 94 regions over the period 1995-2000. The research findings suggest a positive relationship between income and educational inequalities. Fleisher et al. (2011) examine the role of education on worker productivity and firms’ total factor productivity on the basis of firm-level data from China. The study shows that an additional year of schooling raises marginal product by 30.1 %, and the CEO’s education increases TFP for foreign-invested firms. The return is also closely related to ownership. For instance, the effect of schooling on productivity is highest in foreign-invested firms. A significant conclusion is that market mechanisms contribute to a more efficient use of human capital within firms. Zhu (2011)
studies the individual heterogeneity in return to education in China from 1995-2002. The study provides heterogeneous effects both within and between gender groups. Zhu finds that the heterogeneity in schooling returns falls from 1995 to 2002 for both genders in urban China, although their rates of education return have increased substantially. A reason for the narrowing heterogeneity is due to better functioning and an increasingly integrated urban labour market in China.

There is also a large amount of theoretical literature on endogenous knowledge and economic growth. The literature has continued to expand since Romer (1986) re-examined issues of endogenous technological change and economic growth in his 1986 paper (see also, Lucas, 1988; Grossman and Helpman, 1991; and Aghion and Howitt, 1998). But it is the work by Lucas (1988) that has caused a great interest in formal modelling of education and economic growth among economists. The first formal dynamic growth model with education was proposed by Uzawa (1965). Although these studies provide some important insights into relations between economic growth and education, a main problem in the Uzawa-Lucas model and many of their extensions and generalizations is that all skills and human capital are due to formal schooling. However, much of human capital may be accumulated in the family as well as many other social and economic activities. For instance, the human capital of a graduate student from a rich family in the US may be quite different from the human capital of a graduate student from a middle class family in India. Ignoring non-school factors may make us misunderstand the role of formal education in economic development. In addition to formal schooling, this study takes account of Arrow’s learning by doing (Arrow, 1962) and Zhang’s creative leisure (Zhang, 2007) in modelling human capital accumulation.

Another issue in modelling education and economic growth is described by Chen and Chevalier (2008): “Making and exploiting an investment in human capital requires individuals to sacrifice not only consumption, but also leisure. When estimating the returns to education, existing studies typically weigh the monetary costs of schooling (tuition and forgone wages) against increased wages, neglecting the associated labor/leisure tradeoff.” This study makes time distribution between leisure, labour and education endogenous variables. It should also be pointed out that in almost all formal models of economic growth and education the population is assumed to be homogenous. Obviously this is an extremely strict assumption.
This study introduces heterogeneous households into the two-sector growth model with education. Different households have different propensities to save and to receive education, and have different abilities to absorb knowledge and increase human capital through education, learning by doing, and learning by consuming. Most of the extensions and generalizations of the Uzawa-Lucas model are limited to a single representative household. There are a few models of endogenous human capital with heterogeneous households. For instance, Galor and Zeira (1993) propose a model to study the relationship between growth and inequality with human capital as the driving force of economic growth. A main conclusion of their study is that in the presence of credit constraints on human capital investment high initial inequality may reduce long-run growth, while redistribution may increase the growth rate. Maoz and Moav (1999) build a similar model and show that the impact of income redistribution is situation-dependent. In another model by Galor and Moav (2004) it is demonstrated that at the early stage of modern economic development high inequality encourages growth as the rich have a higher propensity to save, whereas at later stages high inequality may discourage growth as human capital becomes increasingly important and high inequality may be an impediment to human capital accumulation. Fender and Wang (2003) build an overlapping-generations model with endogenous education choice with credit or without credit constraints. In their model credit constraints are associated with lower education and a lower rate of interest. Laitner (2000) examines the dynamics of earnings within education groups and overall productivity with a model with endogenous human capital and a distribution of natural abilities. In a model of education where the distribution of abilities is the source of heterogeneiy, Cardak (2004) shows that private education results in higher incomes and less income inequality than in the public education model. Erosa et al. (2010) build a model of endogenous human capital accumulation with education to explain the variation in per capita income across countries. Heterogeneous households make investments in schooling quantity and quality. Further literature can be found in the studies cited. A main deviation of our approach from the previous models is that we derive demand of education in an alternative approach to the typical Ramsey approach. This allows us to explicitly derive the differential equations of the economic system and simulate transition processes. Moreover, we also include endogenous wealth accumulation in our model.
Our model is built upon the Walrasian general equilibrium theory and the three main growth models in the growth literature – the neoclassical two-sector growth model, Arrow's learning by doing model, and the Uzawa-Lucas growth model with education. The main mechanisms of economic structure and growth in these theories are integrated into a single framework. It is also based on the growth model with heterogeneous groups and education by Zhang (2013), and the growth model with economic structure and endogenous time by Zhang (2005). The paper is organized as follows. Section 2 introduces the basic model with wealth accumulation and human capital accumulation for an economy with heterogeneous households. Section 3 examines the dynamic properties of the model and simulates the model with three types of household. Section 4 presents the comparative dynamic analysis. Section 5 concludes the study.

2. THE BASIC MODEL

The economy consists of three sectors: education, capital goods, and consumer goods. Most aspects of the production sectors are similar to the standard two-sector growth model by Uzawa (Uzawa, 1965; Burmeister and Dobell 1970; Azariadis, 1993; and Barro and Sala-i-Martin, 1995). The education sector is based on the Uzawa-Lucas two-sector models (Uzawa, 1965; Lucas, 1988). Households own assets of the economy and distribute their incomes to receive education, to consume and to save. Firms use labour and physical capital inputs to supply goods and services. Exchanges take place in perfectly competitive markets. Factor markets work well and the available factors are fully utilized at every moment. Only households undertake saving. All earnings of firms are distributed in the form of payments to factors of production, labour, managerial skill, and capital ownership. The population is classified into $J$ groups. Each group has a fixed population, $\bar{N}_j$, ($j = 1, ..., J$). Let prices be measured in terms of the commodity and the price of the commodity be unit. Let $p_r(t)$ denote the price of consumer good at time $t$. We denote wage and interest rates by $w_j(t)$ and $r(t)$, respectively. We use $H_j(t)$ to stand for group $j$’s level of human capital.
We use subscript index, $i$, $s$, and $e$, to stand for capital goods, consumer good, and education sectors, respectively. We use $N_m(t)$ and $K_m(t)$ to stand for the labour force and capital stocks employed by sector $m$. Let $T_j(t)$ and $T_{je}(t)$ stand for, respectively, the work time and study time of a typical worker in group $j$. The variable $N(t)$ represents the total qualified labour force. A worker’s labour force is $T_j(t)H_j^{m_j}(t)$, where $m_j$ is a parameter measuring utilization efficiency of human capital by group $j$. The labour input is the work time by the effective human capital. A group’s labour input is the group’s population by each member of the labour force, that is, $T_j(t)H_j^{m_j}(t)N_j$. As the total qualified labour force is the sum of all the groups’ labour forces, we have $N(t)$ as follows

$$N(t) = \sum_{j=1}^{J} T_j(t)H_j^{m_j}(t)N_j, \quad j = 1, ..., J.$$  

(1)

### 2.1 Full employment of labour and capital

The total labour force is employed by the three sectors. The condition of full employment of labour force implies

$$N_i(t) + N_s(t) + N_e(t) = N(t).$$  

(2)

The total capital stock $K(t)$ is allocated between the three sectors. As full employment of capital is assumed, we have

$$K_i(t) + K_s(t) + K_e(t) = K(t).$$  

(3)

Let $\bar{k}_j(t)$ denote per capita wealth of group $j$ at $t$. Group $j$’s wealth is $\bar{k}_j(t)N_j$. As wealth is owned by the households, we have

$$K(t) = \sum_{j=1}^{J} \bar{k}_j(t)N_j.$$  

(4)
2.2 The capital goods sector

We now describe the behaviour of the three sectors. For the two production sectors, we use the neoclassical production functions. Let $F_m(t)$ stand for the production function of sector $m$, $m = i, s$. The production function of the capital goods sector is specified as follows

$$F_i(t) = A_i K_i^{a_i}(t) N_i^{b_i}(t), \quad \alpha_i, \beta_i > 0, \quad \alpha_i + \beta_i = 1,$$

(5)

where $A_i$, $\alpha_i$, and $\beta_i$ are positive parameters. The capital goods sector employs two input factors, capital and labour force. We assume that all the markets are perfectly competitive. The marginal conditions for the capital goods sector are

$$r(t) + \delta_k = \frac{\alpha_i F_i(t)}{K_i(t)}, \quad w(t) = \frac{\beta_i F_i(t)}{N_i(t)}.$$

(6)

2.3 The consumer goods sector

The production function of the consumer goods sector is specified as follows

$$F_s(t) = A_s K_s^{a_s}(t) N_s^{b_s}(t), \quad \alpha_s + \beta_s = 1, \quad \alpha_s, \beta_s > 0,$$

(7)

where $A_s$, $\alpha_s$, and $\beta_s$ are the technological parameter of the service sector. The marginal conditions are given by

$$r(t) + \delta_k = \frac{\alpha_s p_s(t) F_s(t)}{K_s(t)}, \quad w(t) = \frac{\beta_s p_s(t) F_s(t)}{N_s(t)}.$$

(8)

2.4 Education sector

The education sector is the same as in Zhang (2013). It is characterized by perfect competition. Students pay the education fee $p_e(t)$ per unit of time. The education sector pays teachers and capital at the market rates. The total education service is measured by the total education time received by the population. We specify the production function of the education sector as follows
where $A_e$, $\alpha_e$ and $\beta_e$ are positive parameters. Profit is

$$p_e(t)F_e(t) - (r(t) + \delta_k)K_e(t) - w(t)N_e(t).$$

The sector chooses labour and capital to maximize the profit. The marginal conditions for the education sector are thus given by

$$r(t) + \delta_k = \frac{\alpha_e p_e(t)F_e(t)}{K_e(t)}, \quad w(t) = \frac{\beta_e p_e(t)F_e(t)}{N_e(t)}.$$ 

The demand for labour force from the education sector increases in the price and level of human capital and decreases in the wage rate.

### 2.5 Consumer behaviours and wealth dynamics

Consumers make the decisions on choice of consumption levels of education, services and commodities as well as on how much to save. There are different models for decisions on education (Becker, 1981; Cox, 1987; Behrman et al. 1982; Fernandez and Rogerson, 1998; Banerjee, 2004; Florida, et al. 2008; Galindez, 2011). In this study, we follow Zhang (2013) in modelling choice of education time. Let $\bar{k}_j(t)$ stand for the per capita wealth of group $j$. We have $\bar{k}_j(t) = \bar{k}_j(t)\bar{N}_j$. Per capita current income from the interest payment $r(t)\bar{k}_j(t)$ and the wage payment $T_j(t)w_j(t)$ is given by

$$y_j(t) = r(t)\bar{k}_j(t) + T_j(t)w_j(t).$$

We call $y_j(t)$ the current income, in the sense that it comes from consumers’ payment for human capital and efforts and consumers’ current earnings from ownership of wealth. The total value of wealth that consumers can use is $\bar{k}_j(t)$. Here we assume that selling and buying wealth can be conducted instantaneously without any transaction cost. The per capita disposable income is given by
\[ \hat{y}_j(t) = y_j(t) + \bar{k}_j(t) = \left(1 + r(t)\right)\bar{k}_j(t) + T_j(t)w_j(t). \]  

(11)

The disposable income is used for saving, consumption, and education. It should be noted that the value, \( \bar{k}_j(t) \), (i.e., \( p(t)\bar{k}_j(t) \) with \( p(t) = 1 \)), in (11) is a flow variable. Under the assumption that selling wealth can be conducted instantaneously without any transaction cost, we may consider \( \bar{k}_j(t) \) as the amount of the income that the consumer obtains at time \( t \) by selling all of his wealth. Hence, at time \( t \) the consumer has the total amount of income equalling \( \hat{y}_j(t) \) to distribute between saving, consumption, and education.

The typical consumer distributes the total available budget between saving \( s_j(t) \), consumption of consumer goods \( c_j(t) \), and education \( p_e(t)T_{je}(t) \). The budget constraint is

\[ p_s(t)c_j(t) + s_j(t) + p_e(t)T_{je}(t) = \hat{y}_j(t) = \left(1 + r(t)\right)\bar{k}_j(t) + w_j(t)T_j(t), \]

(12)

The time constraint for everyone is

\[ T_j(t) + T_{je}(t) + \tilde{T}_j(t) = T_0, \]

(13)

where \( \tilde{T}_j(t) \) is the leisure time of the representative household and \( T_0 \) is the total available time. Substituting (13) into (12) yields

\[ w_j(t)\tilde{T}_j(t) + p_s(t)c_j(t) + s_j(t) + p_e(t)T_{je}(t) = \bar{y}_j(t) = \left(1 + r(t)\right)\bar{k}_j(t) + T_0w_j(t), \]

(14)

where

\[ p_j(t) = p_e(t) + w_j(t). \]

Following Zhang (2013), we introduce education into the utility function. As education increases human capital, a rise in education tends to result in higher wages (e.g., Heckman, 1976; Lazear, 1977; Malchow-Møller, et al. 2011). As Lazear (1977: 570) points out: “education is simply a normal consumption good
and ..., like all other normal goods, an increase in wealth will produce an increase in the amount of schooling purchased. Increased incomes are associated with higher schooling attainment as the simple result of an income effect.” Education also brings about direct pleasure, greater knowledge, higher social status, and so on. We assume that the consumer’s utility function is dependent on $T_j(t)$, $T_{je}(t)$, $c_j(t)$, and $s_j(t)$ as follows:

\[
U(t) = T_j^{\sigma_{j0}}(t)T_{je}^{\eta_{j0}}(t)c_j^{\xi_{j0}}(t)s_j^{\lambda_{j0}}(t), \quad \sigma_{j0}, \xi_{j0}, \lambda_{j0}, \eta_{j0} > 0,
\]

(15)

where $\sigma_{j0}$ is the propensity to use leisure time, $\eta_{j0}$ the propensity to obtain education, $\xi_{j0}$ the propensity to consume, and $\lambda_{j0}$ the propensity to own wealth. This utility function proposed by Zhang (1993) is applied to different economic problems. Maximizing $U_j(t)$ subject to (9) yields

\[
\begin{align*}
\bar{T}_j(t) &= \frac{\sigma_j \bar{y}_j(t)}{w_j(t)}, \\
T_{je}(t) &= \frac{\eta_j \bar{y}_j(t)}{p_j(t)}, \\
c_j(t) &= \frac{\xi_j \bar{y}_j(t)}{p_s(t)}, \\
s_j(t) &= \lambda_j \bar{y}_j(t),
\end{align*}
\]

(16)

where

$$
\sigma_{j0} = \rho_j \sigma_{j0}, \quad \eta_{j0} = \rho_j \eta_{j0}, \quad \xi_{j0} = \rho_j \xi_{j0}, \quad \lambda_{j0} = \rho_j \lambda_{j0}, \quad \rho_j = \frac{1}{\sigma_{j0} + \xi_{j0} + \lambda_{j0} + \eta_{j0}}.
$$

According to the definitions of $s_j(t)$, the wealth accumulation of the representative household in group $j$ is given by

\[
\tilde{k}_j(t) = s_j(t) - \bar{k}_j(t).
\]

(17)

This equation simply states that the change in wealth is equal to savings minus dissaving.
2.6 Dynamics of human capital

We assume that there are three sources of improving human capital, through education, ‘learning by producing’, and ‘learning by leisure’. Arrow (1962) first introduced learning by doing into growth theory; Uzawa (1965) took account of trade-offs between investment in education and capital accumulation, and Zhang (2007) introduced impact of consumption on human capital accumulation (via ‘creative leisure’) into growth theory. We propose that human capital dynamics is given by

\[ \dot{H}_j(t) = \frac{\nu_{je} F_e^{a_{je}} (t) (H_j^{m_{je}} (t) T_{je} (t) \bar{N}_j)}{H_j^{\pi_{je}} (t) \bar{N}_j} + \frac{\nu_{ji} F_i^{a_{ji}} (t)}{H_j^{\pi_{ji}} (t) \bar{N}_j} + \frac{\nu_{jh} C_{j}^{a_{jh}} (t)}{H_j^{\pi_{jh}} (t) \bar{N}_j} - \delta_{jh} H_j(t), \] (18)

where \( \delta_{jh} (> 0) \) is the depreciation rate of human capital, \( \nu_{je}, \nu_{ji}, \nu_{jh}, a_{je}, b_{je}, a_{ji}, \) and \( a_{jh} \) are non-negative parameters. The signs of the parameters \( \pi_{je}, \pi_{ji}, \) and \( \pi_{jh} \) are not specified as they may be either negative or positive.

As explained in Zhang (2013), the above equation is a synthesis and generalization of Arrow’s, Uzawa’s, and Zhang’s ideas about human capital accumulation. The term

\[ \frac{\nu_{je} F_e^{a_{je}} (t) (H_j^{m_{je}} (t) T_{je} (t) \bar{N}_j)}{H_j^{\pi_{je}} \bar{N}_j} \]

describes the contribution to human capital improvement through education (e.g., Uzawa, 1956; Lucas, 1988; Barro and Sala-i-Martin, 1995, Solow, 2000). Human capital tends to increase with an increase in the level of the education service, \( F_e \), and in the (qualified) total study time, \( H_j^{m_{je}} T_{je} \bar{N}_j \). The population \( \bar{N}_j \) in the denominator measures the contribution in terms of per capita. The term \( H_j^{\pi_{je}} \) indicates that as the level of human capital of the population increases it may be more difficult (in the case of \( \pi_{je} \) being large) or easier (in the case of \( \pi_{je} \) being small) to accumulate more human capital via formal education. We take account of learning by producing effects in human capital accumulation by the term
\( v_{ji} F_j^{a_j} / H_j^{a_j} \). We take account of learning by consuming by the term \( v_{jkh} C_j^{a_j} / H_j^{a_j} \). This term can be interpreted similarly as the term for learning by producing.

### 2.7 Demand for and supply of education

The demand for education from a group is \( T_{je}(t) \bar{N}_j \) and the supply of education service is \( F_e(t) \). The condition that the demand for and supply of education balances at any point in time implies

\[
\sum_{j=1}^{J} T_{je}(t) \bar{N}_j = F_e(t). \tag{19}
\]

### 2.8 Demand for and supply of consumer goods

Only households consume the output of the consumer goods sector. The demand for consumer goods from a group is \( c_j(t) \bar{N}_j \). As the supply is \( F_s(t) \), the condition that the total demand is equal to the total supply implies

\[
\sum_{j=1}^{J} c_j(t) \bar{N}_j = F_s(t). \tag{20}
\]

### 2.9 Demand for and supply of capital goods

As output of the capital goods sector is used only as capital goods, the output should equal the depreciation of capital stock and the net savings. This equilibrium is given by

\[
\sum_{j=1}^{J} s_j(t) \bar{N}_j - K(t) + \delta_k K(t) = F_c(t). \tag{21}
\]

We have completed the model. The model is structurally general in the sense that some well-known models in economics can be considered as its special cases. For instance, if we fix wealth and human capital and allow the number of types of
households to equal the population, then the model is a Walrasian general equilibrium model. If the population is homogeneous, our model is structurally similar to the neoclassical growth model by Solow (1956) and Uzawa (1961). It is structurally similar to the multi-class models by Pasinetti and Samuelson (e.g., Samuelson, 1959; Pasinetti, 1960, 1974). Obviously, if both human capital and physical capital are constant, the model is a Walrasian general equilibrium model. We now examine the dynamics of the model.

3. THE DYNAMICS AND THEIR PROPERTIES

As the system consists of any number of types of household, its dynamics may be highly dimensional. The following lemma shows that the economic dynamics is represented by $2J$ dimensional differential equations.

3.1 Lemma

The dynamics of the economy is governed by the following $2J$ dimensional differential equation system with $z(t)$, $\{\bar{k}_j(t)\}$, and $\{H_j(t)\}$, where $\{\bar{k}_j(t)\} = (\bar{k}_2(t), \ldots, \bar{k}_J(t))$ and $\{H_j(t)\} = (H_1(t), \ldots, H_J(t))$, as the variables

$$\dot{z}(t) = \Lambda_1(z(t), \{H_j(t)\}, \{\bar{k}_j(t)\}),$$

$$\dot{\bar{k}}_j(t) = \Lambda_j(z(t), \{H_j(t)\}, \{\bar{k}_j(t)\}), \quad j = 2, \ldots, J,$$

$$\dot{H}_j(t) = \Omega_j(z(t), \{H_j(t)\}, \{\bar{k}_j(t)\}), \quad j = 1, \ldots, J,$$

in which $\Lambda_j$ and $\Omega_j$ are unique functions of $z(t)$, $\{\bar{k}_j(t)\}$, and $\{H_j(t)\}$ at any point in time, as defined in the Appendix. For any given positive values of $z(t)$, $\{\bar{k}_j(t)\}$, and $\{H_j(t)\}$ at any point of time, the other variables are uniquely determined by the following procedure: $r(t)$ and $w(t)$ by (A3) $\rightarrow$ $w_j(t)$ by (A4) $\rightarrow$ $p_e(t)$ and $p_s(t)$ by (A5) $\rightarrow$ $\bar{k}_i(t)$ by (A18) $\rightarrow$ $N_i(t)$ and $N_e(t)$ by (A14) $\rightarrow$ $N(t)$ by (A11) $\rightarrow$ $N_s(t)$ by (A8) $\rightarrow$ $\bar{y}_j(t)$ by (A6) $\rightarrow$ $K_m(t)$ (A1) $\rightarrow$ $F(t)$, $F_s(t)$,
and $F_c(t)$ by the definitions $\Rightarrow T_j(t)$, $c_j(t)$, $T_{je}(t)$, and $s_j(t)$ by (16) $\Rightarrow K(t)$ by (3).

We have the dynamic equations for the economy with any number of types of household. The system is nonlinear and is of high dimension. It is difficult to generally analyse the behaviour of the system. To illustrate the motion of the system, we specify the parameters as follows:

$$
\begin{align*}
N_1 &= 10, 
N_2 &= 30, 
N_3 &= 60, 
\quad m_1 = 0.5, 
\quad m_2 = 0.4, 
\quad m_3 = 0.3, 
\quad \xi_{10} = 0.12, 
\quad \xi_{20} = 0.18, 
\quad \xi_{30} = 0.2, 
\quad \lambda_{10} = 0.8, 
\quad \lambda_{20} = 0.75, 
\quad \lambda_{30} = 0.7, 
\quad \eta_{10} = 0.015, 
\quad \eta_{20} = 0.010, 
\quad \eta_{30} = 0.008,
\end{align*}
$$

$$
\begin{align*}
\sigma_{10} &= 0.25, 
\sigma_{20} &= 0.23, 
\sigma_{30} &= 0.2, 
\quad v_{1c} = 0.8, 
\quad v_{2c} = 0.7, 
\quad v_{3c} = 0.5, 
\quad v_{2i} = 2.5, 
\quad v_{2i} = 2, 
\quad v_{3i} = 1.7, 
\quad v_{1h} = 0.8, 
\quad v_{2h} = 0.6, 
\quad v_{3h} = 0.5, 
\quad a_{1c} = 0.3, 
\quad a_{2c} = 0.4, 
\quad a_{3c} = 0.45,
\end{align*}
$$

$$
\begin{align*}
b_{1c} &= 0.5, 
     b_{1c} &= a_{4c}, 
     b_{2c} &= a_{5c}, 
     b_{3c} &= a_{3c} 0.6, 
\quad a_{4c} = 0.15, 
\quad a_{5c} = 0.2, 
\quad a_{3c} = 0.5, 
\quad b_{1h} &= 0.3, 
\quad b_{2h} &= 0.35, 
\quad b_{3h} &= 0.4, 
\quad \pi_{1c} = -0.2, 
\quad \pi_{2c} = -0.15, 
\quad \pi_{3c} = -0.1,
\end{align*}
$$

$$
\begin{align*}
\pi_{1i} &= 0.7, 
\pi_{2i} &= 0.75, 
\pi_{3i} &= 0.8, 
\pi_{1h} &= 0.1, 
\pi_{2h} &= 0.15, 
\pi_{3h} &= 0.2, 
\delta_{1i} &= 0.04, 
\delta_{2i} &= 0.05, 
\delta_{3i} &= 0.06, 
\delta_{1h} &= 0.04, 
\delta_{2h} &= 0.05, 
\delta_{3h} &= 0.06, 
\end{align*}
$$

$$
A_1 = 0.9, 
A_2 = A_3 = 0.8, 
\alpha_i = 0.32, 
\alpha_i = 0.34, 
\alpha_i = 0.37, 
T_0 = 1, 
\delta_k = 0.05. 
$$

Groups - 1, 2 and 3’s populations are respectively - 10, 30 and 60. Group 3 has the largest population. The capital goods and education sectors’ total productivities are respectively 0.9 and 0.8. Group 1, 2 and 3’s utilization efficiency parameters, $m_{ij}$, are respectively 0.5, 0.4 and 0.3. Group 1 utilizes human capital most effectively, group 2 the next, and group 3 least effectively. We call the three groups respectively rich, middle, and poor class (RC, MC, PC). We specify the values of the parameters, $\alpha_{ij}$, in the Cobb-Douglas productions as
approximately equal to 0.3. The RC’s ‘learning by doing’ parameter, \( v_{li} \), is the highest. The returns to scale parameters in ‘learning by doing’, \( \pi_{ji} \), are all positive, which implies that knowledge exhibits decreasing returns to scale in learning by doing. The depreciation rates of physical capital and knowledge are specified at about 0.05. The RC’s propensity to save is 0.8 and the PC’s propensity to save is 0.7. The value of the MC’s propensity sits between the other two groups. The RC’s propensity to obtain education is highest among the three classes; the PC has the lowest propensity to obtain education. In Figure 1 we plot the motion of the system with the following initial conditions

\[
z(0) = 0.09, \quad \bar{k}_2(0) = 3, \quad \bar{k}_3(0) = 2, \quad H_1(0) = 9.3, \quad H_2(0) = 3.2, \quad H_3(0) = 1.4. \quad (23)
\]

In Figure 1, the national output is defined as

\[
Y(t) = F_i(t) + p_s(t)F_s(t) + p_e(t)F_e(t).
\]

With different initial conditions, the economy experiences different paths of development. Under (23) the national output experiences positive growth over time, even though the national wealth falls. The three groups all increase education time over time. The RC’s and PC’s levels of human capital are reduced, while the PC’s level of human capital is increased. The wage rates change corresponding to the levels of human capital.
We start with different initial states not far away from the equilibrium point and find that the system approaches an equilibrium point. Under (23) we find that the system has a unique equilibrium. The equilibrium values are listed in (24). The RC has the highest human capital and highest wage rate. The RC also spends much more time in education than the other two groups. According to Dur and Glazer (2008), rich people tend to attend college at a higher rate than poor people as the rich people get more benefits from the consumption content of education. The PC spends the longest time working. The RC’s consumption level and wealth are also highest.

\[ r = 0.05, \ p_s = 1.09, \ p_e = 1.03, \ N = 56.95, \ K = 295.8, \ F_1 = 14.8, \ F_2 = 64.8, \]

\[ F_3 = 1.48, \ N_1 = 9.94, \ N_2 = 46.1, \ N_3 = 0.95, \ K_1 = 47.8, \ K_2 = 242.3, \ K_3 = 5.72, \]

\[ w_1 = 3.1, \ w_2 = 1.6, \ w_3 = 1.2, \ H_1 = 9.4, \ H_2 = 3, \ H_3 = 1.6, \ k_1 = 7.2, \]

\[ k_2 = 3.1, \ k_3 = 2.2, \ T_{1e} = 0.03, \ T_{2e} = 0.02, \ T_{3e} = 0.01, \ T_1 = 0.7, \ T_2 = 0.6, \ T_3 = 0.5, \]

\[ T_1 = 0.25, \ T_2 = 0.38, \ T_3 = 0.45, \ c_1 = 0.99, \ c_2 = 0.68, \ c_3 = 0.58. \]  

(24)
It is straightforward to calculate the six eigenvalues as follows

\[-0.33, -0.30, -0.21, -0.1, -0.07, -0.04.\]

As all the eigenvalues are negative, we see that the equilibrium point is locally stable.

**4. COMPARATIVE DYNAMIC ANALYSIS**

We simulated the motion of the dynamic system. It is important to ask questions such as how a change in one group’s propensity to save or to obtain education affects the economy and each group’s wealth and consumption. First, we examine the case that all the parameters, except the RC’s propensity to obtain education, \( \eta_{10} \), are the same as in (22). We increase the propensity in the following way:

\[ \eta_{10} : 0.015 \Rightarrow 0.018. \]

The simulation results are plotted in Figure 2. In the plots, a variable \( \Delta x_j(t) \) stands for the change rate of the variable, \( x_j(t) \), in percentage due to changes in the parameter value.

**4.1. The rich class’s propensity to receive education being enhanced**

From Figure 2 we see that as the RC increases the propensity to obtain education, the RC’s level of human capital is increased. To examine the process of the change in the human capital, actually we have to follow how all the variables in the system react to the parameter shift. Initially, as the rich class increases its preference for formal education, the rich people will spend more time on schooling (with the other conditions remaining the same). Hence, the human capital of the rich class will be increased. The rise in human capital increases the total labour force, but the fall in work time reduces the labour force. As shown in Figure 2, the total labour force is reduced. Hence, the total output of the industrial sector falls. As the demand for education is increased, the education fee is increased and output of the education sector is increased. The labour share of the education sector is increased. Although the RC increases its education time the education times of the other two classes are slightly affected. The other two classes’ human capital levels are increased, but the PC’s human capital is much less affected. The wage rate of the
RC is increased, but the wage rates of the other two classes are reduced. As the education hours of the MC and PC are slightly affected we can conclude that the returns from schooling in terms of wage rate are reduced for the two classes. As the RC puts more resources into schooling, its wealth and consumption are reduced. It should be noted that in a model with education and inequality by Nakajima and Nakamura (2009), it is found that the educational expenditure of rich households could prevent poor households from escaping poverty. Their model tries to explain the possible effects of high prices of education on growth and inequality in countries, like Japan, Korea, and the U.S. The basic insight from the model is that as the rich households demand higher education, the price will be pushed up. Consequently, the poor are excluded from higher education. This also leads to greater inequality between the rich and poor in the long term. Our model predicts similar effects in a country with heterogeneous households.

Figure 2. A Rise in the Rich Class’s Propensity to Receive Education

4.2 The rich class’s propensity to save being augmented

We now increase the RC’s propensity to save in the following way: \( \lambda_{10} : 0.8 \Rightarrow 0.82 \). The simulation results are plotted in Figure 3. As the RC increases its propensity to save, the wealth per capita of the class is increased. The total capital of the society is increased. As more capital is accumulated in the society, the rate of interest falls. The increase in capital results in increases in wage
rates. As wage rates rise the opportunity costs of education are increased, which initially results in the reduction of education time for the three classes. As the RC accumulates more wealth and the RC’s wage rate is increased, we see that the RC increases its education time in the long term. As the RC reduces education time and puts away more money for future consumption, the RC’s level of human capital falls initially. But in the long term the RC’s human capital is increased as the RC devotes more time to education, becomes more effective through learning by consuming (due to the increased level of consumption) and learning by producing (due to the increased output level of the industrial sector). For the RC the consumption level falls initially, as more disposable income is put away for future consumption. But in the long term the net effects of the change in the propensity increase the consumption level. The other two classes’ levels of consumptions are increased but only slightly. The RC’s wealth is very much increased, but the other two classes’ wealth is only slightly affected. The total labour input is slightly affected. As the RC cares more about wealth accumulation, we see that the share of labour force in the education sector is reduced and that of the industrial sector is increased. It is interesting to note that as the human capital levels of the MC and the PC are increased more than that of the RC and the education fee is also reduced, the MC and PC don’t experience increases in changes in wealth and consumption. We see that a rise in the RC’s propensity to save tends to enlarge the gaps with the other two classes in terms of wealth and consumption levels, but tends to reduce gaps of human capital among the classes. It should be remarked that in this study we fix propensities. In reality, as extremely rich people have ‘too much’ money to spend, the propensity to save tends to increase in the long term. It should be remarked that the propensity to save in our model is different than that defined in the neoclassical growth theory. In our model, the propensity to save is equal to the share of disposable income saved for the future; while in the Solow model the propensity to save is equal to the share of current income saved for the future. Hence, in the Solow model how much wealth people accumulate has a much weaker impact on saving behaviour than in our model, where the wealth per capita is extremely high. It can be seen that our model predicts greater income gaps between the rich and the poor over time, even though the human capital level differences may not vary much between the two groups.
4.3 The rich class’s population being augmented

It has been observed that the effect of population growth varies with the level of economic development and can be positive for some developed economies. Theoretical models with human capital predict situation-dependent interactions between population and economic growth (Ehlich and Lui, 1997; Galor and Weil, 1999; Boucekkine, et al. 2002). We now increase the RC’s population in the following way: $\bar{N}_1 : 10 \Rightarrow 12$. The simulation results are plotted in Figure 4. As the RC increases in population, the RC’s human capital and wage rate fall over time. The schooling time, consumption level, and wealth per capita of the RC rise initially and fall in the long term. The price of education falls, which benefits the MC and PC. As the output of the industrial sector is increased as a consequence of the population increase, the MC and PC learn more effectively through learning by producing. The human capital levels of both the MC and the PC are increased. In the long term the consumption and wealth levels per capita of the MC and the PC are increased. We see that the MC and PC benefit in the long term. We conclude that an increase in the RC’s population reduces the gaps in income, wealth, and consumption between the rich and the poor in the long term.
4.3 The education sector improving the total productivity factor

Another important question is what will happen to different people and the national economy if the total productivity of the education sector is increased. We increase the total productivity in the following way: \( A_e : 0.8 \Rightarrow 0.9 \). The simulation results are plotted in Figure 5. The rise in productivity increases the human capital of all the groups and reduces the price of education. The distribution of the total labour force is slightly affected. The two sectors increase output level in the long term. The education time, wage rates, wealth and consumption levels of all the groups are increased in the long term.
5. CONCLUDING REMARKS

This paper built a growth model of heterogeneous households with endogenous labour supply and human capital accumulation. The model is built on the basis of the well-known Solow-Uzawa neoclassical growth model with endogenous capital and the Uzawa-Lucas two-sector growth model with endogenous human capital. It extended and generalized Zhang’s recent model by adding consumer sector and endogenous choice of leisure time. The economic system consists of three sectors. Human capital is improved in three ways: Arrow’s ‘learning by doing’, Uzawa’s ‘learning by education’, and Zhang’s ‘learning by consuming’. The model describes a dynamic interdependence between wealth accumulation, human capital accumulation, and division of labour, and time distribution among leisure, education, and work under perfect competition. As the model is built for any number of types of household, it is a Walrasian general equilibrium model. We simulated the model for the economy with three groups to demonstrate the existence of equilibrium points and the motion of the dynamic system. We also examined effects of changes in some parameters on the motion of the system. Our comparative dynamic analysis provides some important insights into the interaction between inequality and education. For instance, as the rich class
increases the propensity to obtain education, the human capital of that class increases. But the class’s working hours are reduced, which leads to a fall in the total labour force. The total output of the industrial sector falls. As the demand for education is increased, the education fee is increased and the output of the education sector is increased. The labour share of the education sector is increased. Like the model by Nakajima and Nakamura (2009), our model shows that as rich households demand higher education, the price will be pushed up. Consequently, the poor are excluded from higher education. This also leads to greater inequality between the rich and poor in the long term. We may extend the model in some directions. We may introduce some kind of government intervention in education into the model. In this study we don’t consider public provision or subsidy of education. In the literature of education and economic growth, many models with heterogeneous households are proposed to address issues related to taxation, education policy, distribution of income and wealth, and economic growth (e.g., Bénabou, 2002; Glomm and Kaganovich, 2008).
**APPENDIX: PROVING LEMMA 1**

By (6), (8), and (10) we obtain
\[ z \equiv \frac{r + \delta_k}{w} = \frac{N_q}{\beta_q K_q}, \quad q = i, s, e, \]  
(A1)

where \( \beta_q \equiv \beta_q / \alpha_q \). From (A1) and (3), we obtain
\[ \frac{N_i}{\beta_i} + \frac{N_j}{\beta_j} + \frac{N_k}{\beta_k} = z \sum_{j=1}^{I} k_j N_j, \]  
(A2)

where we also use (4). Insert (A1) in (6)
\[ r(z) = \alpha_r z^{\beta_i} - \delta_k, \quad w(z) = \alpha z^{-\alpha_i}, \]  
(A3)

where
\[ \alpha_r = \alpha_i A_i \beta_i^\beta_i, \quad \alpha = \beta_i A_i \beta_i^{-\alpha_i}. \]

We have
\[ w_j(z, H_j) = H_j^{m_j} w. \]  
(A4)

Hence, we determine the rate of interest and the wage rates as functions of \( z \) and \( (H_j) \). From (7) and (8), and (9) and (10), we have
\[ p_m(z) = \frac{\beta_m^\alpha_m z^{\alpha_m}}{\beta_m A_m}, \quad m = s, e. \]  
(A5)

From (A3) and the definitions of \( \bar{y}_j \), we have
\[ \bar{y}_j = (1 + r)\bar{k}_j + T_0 w_j. \]  
(A6)
Insert $p_s c_j = \xi_j \bar{y}_j$ in (20)

$$\sum_{j=1}^{l} \xi_j \bar{N}_j \bar{y}_j = p_s F_s.$$  \hspace{1cm} (A7)

Substituting (A6) in (A7) yields

$$N_s = \sum_{j=1}^{l} \tilde{g}_j k_j + \tilde{g},$$  \hspace{1cm} (A8)

where we use $p_s F_s = w N_s / \beta_s$ and

$$\tilde{g}_j(z) \equiv r \beta_s \xi_j \bar{N}_j, \quad \tilde{g}(z) \equiv \frac{1 + r}{w}, \quad \tilde{g}(z, (H_j)) \equiv \beta_s T_0 \sum_{j=1}^{l} H_j^m \xi_j \bar{N}_j.$$  \hspace{1cm}

Inserting $w_j T_j = \sigma_j \bar{y}_j$ and $p_j T_{je} = \eta_j \bar{y}_j$ in (13), we have

$$T_j = T_0 - \tilde{p}_j \bar{y}_j,$$  \hspace{1cm} (A9)

where

$$\tilde{p}_j \equiv \frac{\sigma_j}{w_j} + \frac{\eta_j}{p_j}.$$  \hspace{1cm}

Insert (A6) in (A9)

$$T_j = \left(1 - \tilde{p}_j w_j\right)T_0 - \left(1 + r\right)\tilde{p}_j k_j,$$  \hspace{1cm} (A10)

Insert (A10) in (1)

$$N = n_0 - \sum_{j=1}^{l} n_j k_j,$$  \hspace{1cm} (A11)
where

\[ n_0(z, (H_j)) = T_0 \sum_{j=1}^{J} (1 - \tilde{p}_j w_j) H_j^m, \quad n_j(z, (H_j)) = (1 + r) \tilde{p}_j H_j^m. \]

Substituting (A8) and (A11) into (2) and (A2) yields

\[ N_i + N_e = f_n, \]
\[ \frac{N_i}{\beta_i} + \frac{N_e}{\beta_e} = f_k, \quad \text{(A12)} \]

where

\[ f_n(z, (H_j), (\bar{k}_j)) \equiv \tilde{f}_n - (n_i + \tilde{g}_1)\bar{k}_1, \quad \tilde{f}_n(z, (H_j), (\bar{k}_j)) \equiv n_0 - \tilde{g} - \sum_{j=2}^{J} (n_j + \tilde{g}_j)\bar{k}_j, \]
\[ f_k(z, (H_j), (\bar{k}_j)) \equiv \left( zN_i - \frac{\tilde{g}_1}{\beta_i} \right)\bar{k}_1 + f_k - \sum_{j=2}^{J} \left( zN_j - \frac{\tilde{g}_j}{\beta_j} \right)\bar{k}_j - \frac{\tilde{g}}{\beta}, \quad \text{(A13)} \]

where \( \{\bar{k}_j\} \equiv (\bar{k}_2, ..., \bar{k}_J) \). Solve (12) with \( N_i \) and \( N_e \) with the variables

\[ N_i(z, (H_j), (\bar{k}_j)) = \left( \frac{f_n}{\beta_i} - f_k \right)\beta, \quad N_e(z, (H_j), (\bar{k}_j)) = \left( f_k - \frac{f_n}{\beta_e} \right)\beta, \quad \text{(A14)} \]

where

\[ \bar{\beta} \equiv \frac{1}{1/\beta_e - 1/\beta_i}. \]

From (9) and (19), we have

\[ \sum_{j=1}^{J} \eta_j \frac{\bar{N}_j \bar{y}_j}{p_j} = f_e N_e, \quad \text{(A16)} \]
where $T_{je} = \eta_j \bar{y}_j / p_j$ and

$$f_e(z) \equiv A_e \left( \frac{1}{\beta_e z} \right)^{\alpha_e}.$$  

Insert (A6) and (A14) in (A16)

$$(1 + r) \sum_{j=1}^{l} \frac{\eta_j \bar{N}_j k_j}{p_j} = \left( f_k - \frac{f_n}{\beta_i} \right) f_e \bar{\beta} - T_0 \sum_{j=1}^{l} \eta_j w_j \bar{N}_j \frac{p_j}{p_j}. \quad \text{(A17)}$$

Substituting (A13) into (A17) yields

$$\bar{k}_i = \phi(z, \{ k_j \}, (H_j)) \equiv \left[ \left( f_k - \frac{f_n}{\beta_i} \right) f_e \bar{\beta} - T_0 \sum_{j=1}^{l} \eta_j w_j \bar{N}_j \frac{p_j}{p_j} - (1 + r) \sum_{j=2}^{l} \eta_j \bar{N}_j k_j \frac{p_j}{p_j} \right] \phi_0, \quad \text{(A18)}$$

where

$$\phi_0 \equiv \left[ \frac{(1 + r) \eta_1 \bar{N}_1}{p_1} - \left( z \bar{N}_1 - \frac{\tilde{g}_1}{\beta_j} + \frac{n_1 + \tilde{g}_1}{\beta_i} \right) f_e \bar{\beta} \right]^{-1}.$$  

It is straightforward to confirm that all the variables can be expressed as functions of $z$, $\{ k_j \}$ and $(H_j)$ by the following procedure: $r$ and $w$ by (A3) $\rightarrow$ $w_j$ by (A4) $\rightarrow$ $p_e$ and $p_s$ by (A5) $\rightarrow$ $k_1$ by (A18) $\rightarrow$ $N_i$ and $N_e$ by (A14) $\rightarrow$ $N$ by (A11) $\rightarrow$ $N_s$ by (A8) $\rightarrow$ $\bar{y}_j$ by (A6) $\rightarrow$ $K_m$ (A1) $\rightarrow$ $F_i$, $F_s$, and $F_e$ by the definitions $\rightarrow$ $\bar{T}_j$, $c_j$, $T_{je}$, and $s_j$ by (16) $\rightarrow$ $K$ by (3). From this procedure, (A18), (17), and (18), we have

$$\hat{k}_i = \tilde{\phi}_i(z, \{ k_j \}, (H_j)) = \lambda_1 \bar{y}_1 - \varphi, \quad \text{(A19)}$$

$$\hat{k}_j = \Lambda_j(z, \{ k_j \}, (H_j)) = \lambda_j \bar{y}_j - \bar{k}_j, \quad j = 2, \ldots, J,$$
\[ \dot{H}_j = \Omega_j \left( J, \left\{ k_j \right\}, \left( H_j \right) \right), \ j = 1, \ldots, J, \quad (A20) \]

Taking derivatives of equation (A18) with respect to \( t \) and combining with (A20), we get

\[ k_1 = \frac{\partial \varphi}{\partial z} \dot{z} + \sum_{j=2}^{I} \Lambda_j \frac{\partial \varphi}{\partial k_j} + \sum_{j=1}^{I} \Omega_j \frac{\partial \varphi}{\partial k_j}. \quad (A21) \]

Equalling the right-hand sizes of equations (A19) and (A21), we get

\[ \dot{z} = \left[ \Omega_1 - \sum_{j=2}^{I} \Lambda_j \frac{\partial \varphi}{\partial k_j} - \sum_{j=1}^{I} \Omega_j \frac{\partial \varphi}{\partial H_j} \right] \left( \frac{\partial \varphi}{\partial z} \right)^{-1}. \quad (A22) \]

In summary, we proved the lemma.
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**Acknowledgements:** The author is grateful to the constructive comments of the anonymous referee. The author is also grateful for the financial support from the Grants-in-Aid for Scientific Research (C), Project No. 25380246, Japan Society for the Promotion of Science.

*Received: July 05, 2013
Accepted: October 28, 2013*