ABSTRACT: Home equity represents a reserve that can be used for providing additional money for its owners during their retirement. Life insurance models can be successfully applied to model home equity conversion loans. The home equity conversion loan is a financial product that provides a certain flexibility by using home equity as a resource for a quality life during retirement. Home equity conversion loans do not have a predetermined maturity date, as do conventional loans. But, like every loan, it must be repaid. One potential advantage of using a home equity conversion loan during tough financial times instead of some types of need-based assistance is that eligibility is straightforward. Home equity conversion loans can be useful tools in the process of pension system reform.

KEY WORDS: home equity conversion loan, life insurance, actuarial present value, annuity.

JEL CLASSIFICATION: G21, G22
1. INTRODUCTION

Home equity conversion loans are an important retirement planning tool for people who need to use some of the equity in their home for living expenses. A reverse mortgage allows the borrower to tap into his equity without having to sell his home.

Home equity is one of the biggest investments in every person’s lifetime. Also, the process of acquiring a home is, for the average individual, a lifetime process. Therefore it makes sense to use home equity in retirement as a resource for financing the growing cost of medical treatment or for some other purposes.

The primary qualification for a home equity conversion loan is that all borrowers are 65 years of age or older.

Another key eligibility requirement for a home equity conversion loan is that the borrower owns his home. There are limits on how much the borrower may borrow through a reverse mortgage. The amount will depend on the amount of equity that the borrower has in his home and on his age. The older the borrower is the higher is his borrowing limit. Lenders are careful to set reasonable limits, because they depend on the future value of the borrower’s home for repayment of the loan. With a home equity conversion loan the borrower can never owe more than the value of his home.

Life insurance models can be successfully used for home equity conversion loan model creation.

Flexibility of disbursement or payment options is one of the good things about home equity conversion loans. In this paper we will present models for lump sum and monthly payments.

Home equity conversion loans do not have a predetermined maturity date, as do conventional mortgages. But it is a loan, and every loan must be repaid.

Insurance models are used as the basis for home equity conversion loan model construction. There are two types of whole life insurance. The first is term
insurance, where the sum is payable if death occurs within a certain period of time. The second type is deferred whole life insurance. This type of insurance provides a benefit only if the insured survive a certain period of time. It is possible to combine these two types of insurance. In that case we could talk about deferred term insurance, where the sum is payable only if death occurs within a certain period of time, but only if the insured has survived a certain period of time before that period of time.

One potential advantage to using a home equity conversion loan during tough financial times instead of some types of need-based assistance is that eligibility is straightforward.

2. HOME EQUITY CONVERSION LOAN

A home equity conversion loan allows homeowners aged 65 and over to convert some of their home equity into cash without selling or moving out. Homeowners with a home equity conversion loan remain living in their homes. So a home equity conversion loan could be a useful tool in a situation where life expectancy is constantly increasing (Ezra, Collie, & Smith, 2009, p. 22).

Home equity conversion loans are far easier to qualify for than other types of loans. There is no expectation to repay a home equity conversion loan while that equity (house, apartment) is someone’s place of residence.

There are a few basic eligibility requirements:

• borrower must be age 65 or older,
• borrower must own a qualifying property,
• borrower must live in that property as a primary residence.

The primary qualification for a home equity conversion loan is that all borrowers are 65 years of age or older. If the borrower owns his home with a younger spouse, that spouse also needs to be at least 65 in order to qualify for a home equity conversion loan. So, the element of time is important. The element of time refers to the duration of the holding period. In other words, we can look
at a home equity conversion loan as an investment property and the holding period is the length of time a particular piece of investment property is intended to be held (Berges, 2004).

Age also plays another role in home equity conversion loans. Life expectancy is one of the factors lenders use to determine how much to loan to a potential borrower. The older the borrower is the more moneylenders will loan, because the term of the loan will probably be shorter. We could say that the best age for borrowers is somewhere between 72 and 75 years. At that age the borrower is old enough to receive a high loan amount and will probably live in his home long enough to justify the costs.

Another key eligibility requirement for a home equity conversion loan is that the borrower owns his home. Also, there cannot be any liens against the property after closing.

The last of the main eligibility requirements is that the property must be the borrower’s primary residence and not a second home.

One potential advantage of using a home equity conversion loan during tough financial times instead of some types of need-based assistance is that eligibility is straightforward. The proceeds from a home equity conversion loan can be used for any purpose. The funds can be used for basic living expenses, health care, or home improvements, but they also may be used for travel or luxury items.

Home equity conversion loans do come with high fees and costs. On the other hand, almost all of the fees and costs of a home equity conversion loan can be paid from the loan proceeds.

The origination fee covers the lender’s cost of preparing paperwork and processing the loan. Because the origination fee is tied to the lender’s loan preparation costs, it may vary quite a bit among lenders. It may also be negotiable.
The monthly servicing fee covers the lender's cost of maintaining the loan after closing.

Borrowers may be charged some kind of a mortgage insurance premium. The potential purpose for this insurance is to reduce the risk of loss to the lender in the event that the outstanding balance of the loan exceeds the value of the property at the time the mortgage becomes due and payable.

Closing refers to the final steps necessary to complete the loan transaction, including signing all of the legal documents and securing the mortgage. The closing often takes place in person at a bank or title company. To close a loan requires the services of many people other than the lender. For example, different people conduct the appraisal.

2.1 Disbursement Options

Flexibility of disbursement or payment options is one of the good things about home equity conversion loans. The borrower can select a payment plan for the loan proceeds from various options. In this paper we will present models for lump sum and monthly payments. It is possible to talk about some other payment options, like credit line or some combination of lump sum, monthly payments, and credit line, but in this paper we will focus on two basic models.

A lump sum allows the borrower to receive the entire principal loan amount at closing. Monthly payments can be structured as equal monthly payments for life or equal monthly payments for a fixed period of months.

2.2 Repayment and Termination

With a home equity conversion loan the borrower does not have to make monthly payments to repay the loan. Home equity conversion loans do not have a predetermined maturity date, as do conventional loans. But it is a loan, and every loan must be repaid. Home equity conversion loans do not need to be repaid until the last surviving borrower dies. Also, as long as the last surviving borrower lives in the home and uses it as a primary residence, there is no need for the loan to be repaid. It is possible that the borrower defaults or triggers the
acceleration clause under the loan agreement. In that case the loan must be repaid.

When the loan does become due and payable, the entire loan balance must be paid off in full. Most borrowers or their heirs sell the home to repay the loan. It is possible, however, to repay the loan with other funds.

As with any other asset, the borrower can leave the net equity in his home to his heirs. Once the borrower’s home is sold, and his debt to the lender of the reverse mortgage has been paid, any remaining funds are a part of his estate, which he can leave to family members.

3. LIFE INSURANCE – GENERAL MODEL

Based on their distinguishing characteristics, it is possible to identify six distinct types of life insurance contract ( Vaughn & Vaughan, 2008, p. 232 ), but to define all six of them is beyond the scope of this paper.

Initial time $t=0$ is the time of policy issue, and the symbol $\Psi$ stands for time of benefit payment. At that moment the insured risk is going to be realized. Let $X$ be (random) lifetime, or the age-at-death, of the particular individual. On the other hand, realization of the contingencies of life insurance may or may not involve the death of the insured person, but $\Psi$ is a random variable and that has to be taken as fact. In the case of the life insurance for person of age $x$, the random variable $\Psi$ may coincide with the moment of death $T=T(x)$, or it may differ from $T$. The relation between $\Psi$ and $T$ defines the type of insurance product.

Life insurance benefit can be paid at the moment of death. Also, the benefit can be paid at the end of the year of death (or at the end of $m$-thly period), then the payment time is $K+1$ (or $K + \frac{1}{m}$ ), where the curtate time $K=[T]$, is the integer part $T$. 
It is known that life insurance mathematics has its background in probability theory (probability that a certain person will die at some age, or probability that some person will be alive at some point in future). Also, it is known that life insurance mathematics has some background in the theory of interest. The main feature of any life insurance contract consists of the moment of policy issue and the moment of benefit payment. The time lag is determined by two moments. One moment can be determined as the moment of death, but it also can be determined as the payment moment. These two moments can, but also must coincide. We can talk about moment of death $T$ and moment of payment $\Psi$. So, it is necessary to include a present value of the payment in perspective. If we consider the inconsistent financial market, then the present value of the payment of a unit of money is given with a discount factor $v$ (Paramenter, 1999, p. 10). If we look at a situation where $\Psi$ assumes a value $t$, then our present value of the payment of a unit of money is given by $v^t$ for an inconsistent market (Capinski & Zastawniak, 2003, p. 22). In the condition of a consistent market the present value is defined in a similar way. In that case we have continuous time and we assume that interest is compounded continuously. It is also necessary to use the discount factor for a certain period of time, but now the discount factor is defined slightly differently. In that case interest calculation is based on the discount factor that is given by (Gerber, 1997):

$$e^{-\delta t} \quad (0.1)$$

Note that a case where compounding occurs a finite number of times per year is referred to as discrete compounding, while if $n \to \infty$ it is referred to as continuous compounding (Hoy, Livernois, McKenna, Ress, & Stengos, 2001). So, by relation (0.1) we have defined the present value of the payment of a unit of money after $t$ years, where the financial market is consistent and $\delta$ is the unit-time-interval force of interest (Merton, 1990). This approach allows analyzing the case where the interest rate can change more often than once every year (Moller & Steffensen, 2007, p. 49).
We presuppose that the interest rate (or force of interest) is certain (non-random) and does not change in time, or so-called ‘classical actuarial discounting’ (Mario, Bühlmann, & Furrer, 2008, p. 13).

Let us observe a life insurance policy issued by a certain insurance company. The payment follows the death of the insured whenever the death occurs. His inheritors will receive that payment of a unit of money. This type of product is called whole life insurance. As we have said before, \( \Psi \) stands for time of benefit payment. Age-at-death is the random variable. The insurance company must determine the present value of the payment of a unit of money which follows the death of the insured at an unknown moment, where the financial market is consistent with force of interest \( \delta \). The difference between the present value of the payment and the payment itself depends on the force of interest and the time lag between the policy issue and the moment of payment. Force of interest is given as the market value. We already have assumed that force of interest is not random. On the other hand, the death of the insured can occur at any time in the future with certain probabilities. The probability depends on the age of the insured and various other factors. If the moment of payment \( \Psi \) is random, then the present value of the future payment of a unit of money is also random, and by (0.1) is equal to the random variable \( Z \). Random variable \( Z \) could be written as follows:

\[
Z = e^{-\delta \Psi}.
\]  

(0.2)

We can only define the expected value of \( Z \) because \( \Psi \) is a random variable. The expected value for random variable \( Z \) is given as follows:

\[
A = E\{Z\} = E\{e^{-\delta \Psi}\}.
\]  

(0.3)

The quantity \( A \) represents actuarial present value. Actuarial present value represents both segments of the actuarial perspective on life insurance as a two-sided problem. On one side there is the problem of defining the probability that some person will die at age \( x \).
The second side of the problem is defining the present value of the sum payable at some moment in the future. An important fact is that the actuarial present value is the moment-generating function of $\Psi$ (Rotar, 2007), which is given by

$$A = E\{e^{-\delta \psi}\} = M_{\psi}(-\delta).$$

(0.4)

By $Z = e^{-\delta \psi}$, the $l$th moment is

$$E\{Z^l\} = E\{e^{-l\delta \psi}\} = M_{\psi}(-l\delta).$$

(0.5)

If we know the moment-generating function $M_{\psi}(s)$, then we know all moments of $Z$. So, for $l=1$ we have

$$E\{Z\} = E\{e^{-\delta \psi}\} = M_{\psi}(-\delta),$$

(0.6)

and for $l=2$:

$$E\{Z^2\} = E\{e^{-2\delta \psi}\} = M_{\psi}(-2\delta).$$

(0.7)

Now, it is possible to define variance for our random variable:

$$\text{Var}\{Z\} = E\{Z^2\} - \left(E\{Z\}\right)^2 = M_{\psi}(-2\delta) - \left(M_{\psi}(-2\delta)\right)^2.$$ $$\text{Var}\{Z\} = E\{Z^2\} - \left(E\{Z\}\right)^2 = M_{\psi}(-2\delta) - \left(M_{\psi}(-2\delta)\right)^2.$$ (0.8)

Let $\mu(x)=\mu$, where $\mu$ stands for force of mortality. Then $X$ is exponential, and by the lack-of-memory property $T(x)$ is also exponential (Ross, 1997).

If the random variable has a distinct moment-generating function, then the random variable has a distinct distribution. Also, if the random variable has a distinct distribution, then the random variable has a distinct moment-generating function (Epps, 2009). It is known that the moment-generating function for an exponential random variable with the parameter $a$ and for $z<a$ is (Rotar, 2007)
\[ M(z) = \int_0^\infty e^{zx}e^{-ax}dx = \frac{a}{a-z} = \frac{1}{1-z/a}. \]  

(0.9)

For \( Z \geq a \) a moment-generating function does not exist. Therefore, we could conclude that the moment-generating function for random variable \( T(x) \) is given by \( M_{T(x)}(z) = \frac{\mu}{\mu - z} \). So, by (0.3) and (0.9) we have as follows:

\[ \bar{A}_x = \frac{\mu}{\mu + \delta} \]  

(0.10)

and

\[ \text{Var}(Z) = \frac{\mu}{\mu + 2\delta} - \left( \frac{\mu}{\mu + \delta} \right)^2. \]  

(0.11)

It is obvious that to calculate the expected value of actuarial present value and its variance it is not necessary to use age-at-death \( T(x) \). It is enough that we know force of mortality and force of interest.

### 3.1 Whole life insurance – benefits payable at the moment of death

In this case, \( \Psi = T = T(x) \) for a life-age \( x \). Then the present value of the payment is \( Z = e^{-\delta T(x)} \), and the actuarial present value is

\[ \bar{A}_x = E\{e^{-\delta T(x)}\} = M_{T(x)}(-\delta). \]  

(0.0.12)

In (0.0.12) \( M_{T(x)} \) is the moment-generating function of \( T(x) \). The density of \( T(x) \) (Klugman, Panjer, & Willmot, 2004) is given as follows:

\[ f_{T(x)}(t) = \mu_x(t) \cdot p_x = \mu(x + t) \cdot p_x. \]  

(0.0.13)
According to (0.0.12) and (0.0.13), it follows that

$$\bar{A}_x = \int_0^\infty e^{-\delta t} \mu_x(t) p_x dt.$$  \hfill (0.0.14)

We could see that for the exponential random variable the force of interest is constant and $$\bar{A}_x = \frac{\mu}{\mu + \delta}.$$ 

As we saw, $$E\{Z^2\} = E\{e^{-2\delta T(x)}\},$$ which corresponds to the actuarial present value for the doubled interest rate and is given by $$2\bar{A}_x = E\{Z^2\} = E\{e^{-2\delta T(x)}\}.$$ Making use of this notation, we can write:

$$\text{Var}\{Z\} = 2\bar{A}_x - (\bar{A}_x)^2.$$ \hfill (0.0.15)

### 3.2. Whole life insurance – benefits payable at the end of the year of death

The models below could be applied for any choice of time unit. The fact that in the title a year has been chosen as the time unit does not restrict using the models presented below for any time unit. In this case we have $$\Psi = K(x)+1.$$

If we know that equality $$P(K(x) = k) = p_k x \alpha_{x+k}$$ is true, than we can derive the following expression for actuarial present value:

$$A_x = E\{e^{-\delta P}\} = \sum_{k=0}^\infty e^{-\delta(k+1)} P(K(x) = k) = \sum_{k=0}^\infty e^{-\delta(k+1)} \times \frac{p_k x \alpha_{x+k}}{1}.$$ \hfill (0.0.16)

Let us estimate the net premium $$A_{60}$$ for Yugoslav demographic tables 1952–1954 and $$\delta = 0.05$$ (Šain, 2009., p. 282) So we have:

$$P(K(50) = k) = \frac{l_{60+k}}{l_{60}} \frac{d_{60+k}}{l_{60+k}} = \frac{d_{60+k}}{l_{60}}.$$
It makes sense to take \( k = 30 \). It is large enough. We can see that \( q_x = \frac{d_x}{l_x} \) (Paramenter, 1999, p. 128). If we take the expression above and put it in (0.0.16), then it follows that

\[
A_{x0} = \sum_{k=0}^{40} e^{-\delta(k+1)} \frac{d_{60+k}}{l_{60}} = 0.242656.
\]

Another way to compute actuarial present value is based on the following relation:

\[
A_x = e^{-\delta} q_x + p_x e^{-\delta} A_{x+1} = e^{-\delta} (q_x + p_x A_{x+1}). 
\] (0.0.17)

If we know the value of \( A_n \) and \( p_x \) we can move 'backwards' and compute \( A_x \) \((n > x)\). The proof for relation (0.0.17) is intuitive by nature. Let us set some initial moment as the standpoint for the process which has been observed. Starting from that moment, the insured person can either die or survive during the next year. The probability that he will die is \( q_x \). The insured sum is one unit of money and its present value is \( e^{-\delta} \). The probability that he will survive is \( p_x \). In that case the insured person will be \( x+1 \) years old and his life is 'continued'. The insurance process will start over and the present value of the insured sum is \( A_{x+1} \). We can conclude that the actuarial present value is the sum of the two summands. The first is \( e^{-\delta} \) with probability \( q_x \) and the second is \( A_{x+1} \) with probability \( p_x \). So it follows that

\[
E \{ Z \} = E \{ Z | T(x) \leq 1 \} P(T(x) \leq 1) + E \{ Z | T(x) > 1 \} P(T(x) > 1) = E \{ Z | T(x) \leq 1 \} q_x + E \{ Z | T(x) > 1 \} p_x,
\]

only if \( E \{ Z | T(x) \leq 1 \} = e^{-\delta} \times E \{ Z | T(x) > 1 \} = e^{-\delta} A_{x+1} \) is true.
Let us prove that \( \mathbb{E}\{Z|T(x) > 1\} = e^{-\delta} A_{x+1} \) is true. In that case we have:

\[
\mathbb{E}\{Z|T(x) > 1\} = \mathbb{E}\left\{e^{\delta(K(x)+1)}|T(x) > 1\right\} \\
= e^{-\delta} \mathbb{E}\left\{e^{-\delta K(x)}|T(x) > 1\right\} \\
= e^{-\delta} \mathbb{E}\left\{e^{-\delta K(x+1)}|T(x) > 1\right\} \\
= e^{-\delta} \mathbb{E}\left\{e^{-\delta K(x+1)}\right\} = e^{-\delta} A_{x+1}.
\]

We have used the fact that life expectancy for an insured person who has survived the first year is equal to the sum of one year and life expectancy for an \( X+1 \) years-old person expressed in years.

Analogous to the case when benefit is payable at the moment of death, we have the case when benefit is payable at the end of the year of death and force of mortality is constant net premium defined by \( A_x = \frac{q_x}{q_x + i} \) (Batten, 2005). This relation could be written in another way, as follows:

\[
A_x = \frac{q_x}{q_x + i} = \frac{1 - p_x}{1 - p_x + e^{-\delta} - 1} = \frac{1 - p_x}{p_x + e^{-\delta}}.
\]

Let us look at the relation between \( A_x \) and \( \bar{A}_x \) with more attention. The main tool for this relation’s analysis is linear interpolation procedure. Here the assumption is made that lifetime is uniformly distributed within each year of age. In that case we have as follows (Bowers, Gerber, Hickman, Jones, & Nesbit, 1997):

\[
\bar{A}_x = \frac{i}{\delta} A_x,
\] (0.0.18)
where, as we have stated before, \( i = e^\delta - 1 \). So \( i \) is effective annual rate (yield).

It is important to notice that for small \( \delta \), correction coefficient \( \frac{i}{\delta} \to 1 \) (Mario, Bühlamann, & Furrer, 2008). Let us look at the situation where we have \( \delta = 0.04 \). In that case \( i = e^{0.04} - 1 \approx 0.04081 \), and \( \frac{0.04081}{0.04} = 1.02 \). This becomes understandable if we recall that \( e^x = 1 + x + \frac{x^2}{2} + o(x^2) \).

So, if we use that fact we have as follows:

\[
1 \leq \frac{i}{\delta} = \frac{e^\delta - 1}{\delta} = 1 + \frac{\delta}{2} + O(\delta). \tag{0.0.19}
\]

In relation (0.0.19) \( \frac{i}{\delta} \) differs from 1 by approximately \( \frac{\delta}{2} \).

Let us prove that \( \bar{A}_x = \frac{i}{\delta} A_x \). According to that we will observe two quantities, lifetime \( T = T(x) \) and corresponding curtate time \( K = K(x) \). Let us introduce the third quantity. This quantity is defined as the difference between the first two (the bigger one \( T = T(x) \), and the smaller one \( K = K(x) \)), and it could be named as a fractional part of lifetime \( T = T(x) \). We will denote this quantity by \( T_r(x) \). As we have said, \( T_r(x) = T(x) - K(x) \). \( K = K(x) \) and \( T_r(x) \) are independent and \( T_r(x) \) is uniform at \([0,1]\). We could write as follows:

\[
\bar{A}_x = E\left\{e^{-\delta T} \right\} = E\left\{e^{-\delta (T_r + K)} \right\} = E\left\{e^{-\delta K} \right\} E\left\{e^{-\delta T_r} \right\} = e^\delta E\left\{e^{-\delta (K+1)} \right\} E\left\{e^{-\delta T_r} \right\} = e^\delta A_x E\left\{e^{-\delta T_r} \right\}. \tag{0.0.20}
\]
It is known that $E\{e^{-\delta T_r}\} = M_{T_r}(-\delta)$ where $M_{T_r}(-Z)$ is the moment-generating function for $T_r$. Also, we know that $E\{e^{-\delta T_r}\} = \frac{1-e^{-\delta}}{\delta}$ and according to (0.0.20) we could write it as follows:

$$\bar{A}_x = e^{\delta} \frac{1-e^{-\delta}}{\delta} A_x = \frac{e^{\delta}-1}{\delta} A_x.$$  

(0.0.21)

The link between (0.0.21) and (0.0.18) is obvious.

3.3. The case of benefits payable at the end of the $m$-nthly period

Let us denote by $A_x^{(m)}$ the actuarial present value of benefits payable at the $m$-nthly period. So here we will present an approximation formula for $A_x^{(m)}$ assuming that the lifetime is uniformly distributed in each year. According to that we can write (Bowers, Gerber, Hickman, Jones, & Nesbit, 1997)

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x.$$  

(0.0.22)

It is important to mention that in the relation above $i^{(m)} = m\left[(1+i)^{1/m} - 1\right]$.

Relation (0.0.22) follows from (0.0.18) and we know that $\lim_{m \to \infty} i^{(m)} = \delta = \ln(1+i)$. According to that we can conclude that the coefficient of correction $\frac{i}{i^{(m)}}$ is increasing if $m$ is increasing.

Hence, for any $m$:

$$1 \leq \frac{i}{i^{(m)}} < \frac{i}{\delta} = 1 + \frac{\delta}{2} + O(\delta).$$
According to what has been said above, we now can write the relation as follows:

\[
0 \leq \frac{i}{i^{(m)}} - 1 < \frac{i}{\delta} \leq \frac{\delta}{2} + e^2 \frac{\delta^2}{6}.
\]  

(0.0.23)

It is obvious that the coefficient of correction is increasing from 1 to \( \frac{i}{\delta} \).

Let us prove the expression for the actuarial present value where benefit is payable at the end of \( m \)-thly period of the year in which death occurs. Denote by \( K^{(m)} \) the number of complete periods of the length \( 1/m \) that the insured survived. Let us call these periods \( m \)-ths. Set \( R^{(m)} = K^{(m)} - mK \). This is the variable which represents the number of complete \( m \)-ths lived in the year of death. Let us look at a simple example where \( m = 12 \); in other words, we are observing \( \frac{1}{12} \)th part of the year, although it is not true that every month is the same (number of days differ from month to month). Also, let’s say that we are observing the person insured at age \( T = 24.48 \). This age is expressed in years. If we express that age by months we have 293.6. We could write this as follows: \( K^{(m)} = 293 \), and \( mK = 288 \). It is obvious that the person insured has lived 5 months during his 25th year of life and approximately 60% of the 6th month. We can write \( R^{(m)} = 5 \).

Let us come back to the general case. Under the assumption made the random variables \( R^{(m)} \) and \( K \) are independent, and \( R^{(m)} \) takes on values \( 1, 2, \ldots, (m-1) \) with the same probability. Variable \( R^{(m)} \) takes the mentioned values because it is limited and it can be as large as the basic unit (in our case that variable can be related to the 11-month period). We mentioned before that we have taken an annual interest rate. According to that we must take a relative
interest rate (we need to divide our annual interest rate by $(m)$ and then
discount benefit for $(K^{(m)} - 1)$.  

Then we can write:

$$A_x^{(m)} = \mathbb{E} \left\{ \exp \left\{ -\delta \left( \frac{K^{(m)} + 1}{m} \right) \right\} \right\} =$$

$$= \mathbb{E} \left\{ \exp \left\{ -\delta \left( mK + R^{(m)} + 1 \right) \right\} \right\} =$$

$$= \mathbb{E} \left\{ \exp \left\{ -\delta \left( mK + R^{(m)} + 1 \right) \right\} \right\} =$$

$$= \mathbb{E} \left\{ \exp \left\{ -\delta K \right\} \right\} \mathbb{E} \left\{ \exp \left\{ -\delta \left( R^{(m)} + 1 \right) \right\} \right\} =$$

$$= \mathbb{E} \left\{ \exp \left\{ -\delta K \right\} \right\} \mathbb{E} \left\{ \exp \left\{ -\frac{\delta}{m} \left( R^{(m)} + 1 \right) \right\} \right\} =$$

$$= e^{\delta} \mathbb{E} \left\{ \exp \left\{ -\delta (K + 1) \right\} \right\} \mathbb{E} \left\{ \exp \left\{ -\frac{\delta}{m} \left( R^{(m)} + 1 \right) \right\} \right\} =$$

$$= A_x e^{\delta} \mathbb{E} \left\{ \exp \left\{ -\frac{\delta}{m} (R + 1) \right\} \right\}.$$

According to (0.0.12) expression $\mathbb{E} \left\{ \exp \left\{ -\frac{\delta}{m} (R + 1) \right\} \right\}$ can be written down
as $M_{R^{(m)} + 1} \left( -\frac{\delta}{m} \right)$, where $M_{R^{(m)} + 1} (Z)$ is the moment-generating function for
random variable $R^{(m)} + 1$. 

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If we know that the random variable takes values 1, 2, ..., m with identical probabilities, then we can write it as follows:

\[
M_{\mathbf{R}^m+1}(z) = \sum_{k=1}^{m} e^{zk} \frac{1}{m} = \frac{1}{m} e^{z} \frac{1-e^{mnz}}{1-e^{z}}. \tag{0.0.24}
\]

### 3.4. Deferred whole life insurance

This type of insurance product provides a benefit only if the insured survives c years. As we have said before, we set \( \Psi = \infty \) if the condition for benefit payment has not been satisfied. So, for \( T(x) < c \) we have \( \Psi = \infty \), and the opposite, if \( T(x) \geq c \) we have \( T(x) = \Psi \). The case where \( T(x) = c \) is irrelevant, because we are observing the continuous variable. Because \( Z = e^{-\delta \Psi} \),

\[
Z = \begin{cases} 
0 & T(x) \leq c \\
\frac{e^{-\delta T(x)}}{1-e^{-\delta T(x)}} & T(x) > c 
\end{cases} \tag{0.0.25}
\]

Let us observe the situation where death occurs during the period of c years and \( \delta = 0.1 \). In that case the present value of the payment of a unit of money is \( Z = e^{-\delta \Psi} = e^{-0.1 \times \infty} = 0 \). Let us assume that we have no yield on invested capital, or \( \delta = 0 \). So according to that we have \( (\delta = \ln(1+i) = \ln(1+0)) \).

Then, the exponent of Napier’s number is an undefined expression, or \( 0 \times \infty \). However, the exponential function is ‘more powerful’ than the logarithmic function. So, according to that in exponent of Napier’s number we will have \( -\infty \) and then our benefit equals zero.

In this case we will denote actuarial present value by

\[
q_{\mathcal{X}}^\mathcal{X} = q_{\mathcal{X}} p_{\mathcal{X}} e^{-\delta c} \bar{A}_{\mathcal{X}+c}. \tag{0.0.26}
\]

As we can see, the actuarial present value for the deferred whole life insurance can be presented as the product of the actuarial present value for the whole life
insurance with the starting age $X + c$ and the probability that the insured person will survive $c$ years. This product must be discounted for $c$ years.

So we can write (Rotar, 2007):

$$E\{Z\} = 0 \times P(T(x) \leq c) + E\{Z|T(x) > c\} \times P(T(x) > c) =$$

$$= E\left\{ e^{-\delta T(x)} \right\} | T(x) > c \times p_x = c \cdot p_x E\left\{ e^{-\delta (k + T(x + c))} \right\} =$$

$$= c \cdot p_x e^{-\delta c} E\left\{ e^{-\delta T(x + c)} \right\} = c \cdot p_x e^{-\delta c} A_{x+c}.$$  

Let us observe the case where benefit is payable at the end of the period in which death occurs. So,

$$\Psi = \infty \quad \iff \quad T(x) \leq c \implies Z = 0$$

$$\Psi = K(x) + 1 \quad \iff \quad T(x) > c \implies Z = e^{-\delta (K(x) + 1)}.$$  

We will denote the actuarial present value in this case as follows:

$$q A_x = \sum_{c=K}^{\infty} e^{-\delta (k+1)} * P(K(x) = k).$$

Or we could write it down as follows (Rotar, 2007):

$$q A_x = c \cdot p_x e^{-\delta c} A_{x+c}.  \quad \quad (0.0.29)$$

Because the number of discount periods is the integer in this case, we could write it as follows:

$$q A_x = c \cdot p_x e^{c} A_{x+c}.  \quad \quad (0.0.30)$$
3.5. Term insurance – the continuous-time case

This type of insurance product, in the general case, provides payment of the unit of money if death occurs during \( n \) years. So, we can write:

\[
(\Psi = \infty \Leftrightarrow T(x) \leq n) \Rightarrow Z = e^{-\delta T(x)},
\]

\[
(\Psi = T(x) \Leftrightarrow T(x) > n) \Rightarrow Z = 0.
\]

(0.0.31)

In this case we will denote actuarial present value by

\[
\overline{A}_{x+n}^l = \int_0^n e^{-\delta t} f_{T(x)}(t) \, dt.
\]

(0.0.32)

Here, the actuarial present value depends on the distribution of the random variable in the interval \([0, n]\) and it does not depend on the mortality rate after \( n \) years (Rotar, 2007).

It is possible to use the following formula:

\[
\overline{A}_x = \overline{A}_{x+n}^l + e^{-\delta n} p_x \overline{A}_{x+n}.
\]

(0.0.33)

Formula (0.0.33) can be written in another way, as follows:

\[
\overline{A}_{x+n}^l = e^{-\delta n} p_x \overline{A}_{x+n} - \overline{A}_x.
\]

(0.0.34)

Proof for relation (0.0.33) is simple and it follows from relation (0.0.31), and the (0.0.25) equation reference goes here. If we denote by \( Z_i \) the random variable in (0.0.25) where \( C = n \), then \( Z_i \) is the same as in (0.0.31). Let us write:

\[
Z = \begin{cases} 
  e^{-\delta T(x)}, & T(x) \leq n \\
  0, & T(x) > n
\end{cases}, \quad Z_i = \begin{cases} 
  0, & T(x) \leq n \\
  e^{-\delta T(x)}, & T(x) > n
\end{cases}
\]

(0.0.35)
It is obvious that $Z + Z_1 = e^{-\delta T(x)}$ and that the relation in fact presents the present value for a payment of a unit of money for the whole life insurance contract.

Let us look the same insurance product but for the case where benefit is payable at the end of the year (or some other basic time period). So, for term insurance for $n$ years we have:

$$Z = \begin{cases} e^{-\delta (K(x)+1)}, & T(x) \leq n \\ 0, & T(x) > n \end{cases} \quad (0.0.36)$$

In that case we will denote actuarial present value by $A_{x\mid n}$. Also, we can write:

$$A_{x\mid n} = \sum_{k=0}^{n-1} e^{-\delta(k+1)} P(K(x) = k) = \sum_{k=0}^{n-1} e^{-\delta(k+1)} n_p x_A x_k x_k. \quad (0.0.37)$$

In that sense, if we observe (0.0.33) then we have:

$$A_x = A_{x\mid n} + e^{-\delta n} n_p x_A x_{x+n}, \quad (0.0.38)$$

and it is possible to write:

$$A^{\dagger}_{x\mid n} = e^{-\delta n} n_p x_A x_{x+n} - A_x. \quad (0.0.39)$$

### 3.6. Pure endowment

In this case benefit is payable only if the insured survives until a certain age. To be precise, the insured will receive payment if death does not occur within $n$ years. So, it is $\Psi = n$ for $T(x) > n$ and $\Psi = \infty$ for $T(x) \leq n$. We can make the next conclusion:
In this case we will denote the actuarial present value $nE_x$ and it will be defined as follows:

$$nE_x = E\{Z\} = e^{-\delta n}p(T(x) > n) = e^{-\delta n}p_x.$$  \hfill (0.0.41)

If the random variable is exponential (constant force of mortality) we have:

$$nE_x = e^{-\int_0^n [(\mu(x+t)+\delta)dt]}.$$  \hfill (0.0.42)

Relation (0.0.42) indicates that increase of force of mortality has same effect as the increase of force of interest. This works on both sides.

If we know that the random variable can take value 0 or some other exactly determined value with probabilities $q$ and $p$, respectively, than we can determine the variance for the mentioned random variable as follows:

$$\text{Var}\{Z\} = e^{-\delta 2n}p_x(1 - n p_x).$$  \hfill (0.0.43)

### 3.7. Endowment

In this case benefit is payable if the insured survives $n$ years, but also if he dies during the next $n$ years his inheritors will receive payment. If the inheritors receive payment at the exact moment when the insured dies, we can write:

$$Z = \begin{cases} e^{-\delta T(x)}, & T(x) \leq n \\ e^{-\delta n}, & T(x) > n \end{cases}.$$  \hfill (0.0.44)

Let us denote the actuarial present value $\overline{A}_{x\,n\,}\,\cdot$ Also we will denote the random variable in (0.0.31) by $Z_1$, and in (0.0.40) by $Z_2$. So we can write:
\[ Z_1 = \begin{cases} e^{-\delta n}, & T(x) \leq n \\ 0, & T(x) > n \end{cases} \] 

(0.0.45)

and

\[ Z_2 = \begin{cases} 0, & T(x) \leq n \\ e^{-\delta n}, & T(x) > n \end{cases} \] 

(0.0.46)

In this case the present value for this type of insurance product is given by 

\[ Z = Z_1 + Z_2. \]

So,

\[
\bar{A}_{x\mid T} = E\{Z\} = E\{Z_1\} + E\{Z_2\} = \bar{A}_{x\mid T}^1 + n E_x = \bar{A}_{x\mid T}^1 + e^{-\delta n} \cdot p_x.
\] 

(0.0.47)

If we have a case where benefit is payable at the end of the year in which death occurs, we can write as follows:

\[
Z = \begin{cases} e^{-\delta (K(x)+1)}, & K(x) < n \\ e^{-\delta n}, & K(x) \geq n \end{cases}
\] 

(0.0.48)

Here we have two different cases:

1) The first case is when the insured dies during interval \([n-1,n)\) and benefit will be paid to inheritors at the end of the \(n\)-th year. So, \(t = n\) (if we know that \(n = K - 1\) than also \(\min\{K(x)+1, n\} = n\)),

2) The second case is when the insured lives until \(x + n\) years, then benefit is payable at the \(n\)-th moment (we know that \(K(x) \geq n\) and \(\min\{K(x)+1, n\} = n\)).

Let us denote the actuarial present value in this case by \(\bar{A}_{x\mid T}\).
Let us make a parallel with (0.0.45) and (0.0.46): then for benefit payable at the end of the year when death occurs or at the end of the year upon which the insured has survived:

\[ Z_1 = \begin{cases} e^{-\delta(K(x)+1)}, & K(x) < n \\ 0, & K(x) \geq n. \end{cases} \]  

(0.0.49)

Also we can write:

\[ Z_2 = \begin{cases} 0, & K(x) < n \\ e^{-\delta n}, & K(x) \geq n \end{cases} \]  

(0.0.50)

So, we have:

\[ A_{x\lvert n} = A_{x\lvert n}^1 + nE_x = A_{x\lvert n}^1 + e^{-\delta n} nP_x. \]  

(0.0.51)

If we observe expression (0.0.51) in the context where force of interest is double sized, then:

\[ 2A_{x\lvert n} = 2A_{x\lvert n}^1 + 2E_x = 2A_{x\lvert n}^1 + e^{-2\delta n} nP_x. \]  

(0.0.52)

Expression (0.0.52) provides a simple way to calculate variance for our random variable.

If force of mortality is constant, then we have:

\[ nE_x = e^{-r(\mu+\delta)}. \]  

(0.0.53)

If we know that force of mortality is constant, expression (0.0.32) can be defined as follows:

\[ \bar{A}_{x\lvert n}^1 = A_x (1 - nE_x). \]
On the other hand, if we have a similar situation, but where benefit is payable at the end of the year in which death occurs, then (Batten, 2005):

\[
\bar{A}_{x+nIL} = A_x (1 - n E_x).
\]

### 3.8. Annuity models

The original meaning of 'annuity' was an asset that paid annual income, and there was no necessary connection to an individual or group of individuals being alive. An asset with life-contingent payments was called a life annuity to distinguish it from certain annuity (Cannon & Tonks, 2008).

In this paper we will present an untraditional approach to annuity modelling. The connection between actuarial present value and whole life annuities is analogous to the connection between present value and certain annuity-due. Thus, annuity-due with unit rate is given by

\[
\bar{a}_{nIL} = 1 + v + \ldots + v^{n-1} = \frac{1 - v^n}{1 - v} = \frac{1 - \nu^i}{d}
\]

In relations (0.0.54) \( v^n \) is the present value of one unit to be paid after \( n \) periods (years, months, or some other period of time). It is known that the discount factor is defined by \( v = \frac{1}{1+i} \), where \( i \) is the annual interest rate. Also, we can write \( d = \frac{i}{1+i} \), where \( d \) denotes the discount rate or rate of interest-in-advance.

The perpetuity-due (\( \nu < 1 \) and an infinite horizon \( n = \infty \)) is given by

\[
\bar{a}_{\infty} = \frac{1}{1 - \nu}
\]
If payments are made at the end of the periods, than the present value of the cash flow equals $v + v^2 + ... + v^n = \frac{1 - v^n}{1 - v}$. This quantity is denoted by $\overline{a}_n$ and is called an annuity-immediate (or payable in arrears). So, for perpetuity-immediate we can write $\overline{a}_\infty = \frac{v}{1 - v}$ (for $v < 1$). Here we are talking about certain annuities – there are no probability considerations.

It is obvious that $\overline{a}_\infty = v\overline{a}_\infty$.

### 3.8.1. Whole life annuities

Let us consider the whole life annuity-due. The actuarial present value of the annuity is denoted by $\overline{a}_x$. Let us make a parallel with relation (0.0.54). We can put the actuarial present value instead of the present value in relation (0.0.54). So we can put quantity $A_x$ instead of $v^n$. Quantity $A_x$ is given by relation (0.0.16). Also the same quantity is given by relation (0.0.17). We have mentioned earlier that $A_x = \frac{q_x}{q_x + 1}$. Hence, the actuarial present value of whole life annuity-due is given by

$$\overline{a}_x = \frac{1 - A_x}{d}.$$  \hspace{1cm} (0.0.56)

In relation (0.0.56) payments are made at the beginning of each period. If we observe the situation where payments are made at the end of each period (we will denote actuarial present value in this case by $a_x$), then it is obvious that the difference between $\overline{a}_x$ and $a_x$ is the payment that is made at the beginning of the first period. In the general case we are assuming that all payments are equal to the one unit of money. So we can write

$$a_x = \overline{a}_x - 1$$  \hspace{1cm} (0.0.57)
We can prove relation (0.0.57) (Rotar, 2007). Let $\Psi$ be an integer-valued random variable. Consider $\Psi$ time intervals of a unit length, say, $\Psi$ complete years. Denote by $c_i$ the payment at the integer time $t$. If the first payment is made at the initial time $t = 0$, the second payment is made at the beginning of the second period (that is, at the time $t = 1$), and so on, then the present value of total payment during $\Psi$ periods is equal to

$$Y = c_0 + c_1 \nu + c_2 \nu^2 + \ldots + c_{\Psi-1} \nu^{\Psi-1} \quad (0.0.58)$$

So, we have set present values for annuities-due, where the final number of annuities that should be paid is defined as a random variable. That random variable may coincide with the future lifetime $T(x)$.

Let us consider annuities-immediate (or payable in arrears) as well. In that case we have $\tilde{\Psi}$ intervals, and we are assuming that payments are made at the end of each interval. Then the present value of the total payment is

$$\tilde{Y} = c_1 \nu + c_2 \nu^2 + \ldots + c_n \nu^n \quad (0.0.59)$$

In relation (0.0.58) the number of payments is $\Psi - 1$. This is obvious – we have $\Psi$ periods and payments that are provided at the beginning of each interval. Thus, in relation (0.0.59) we have $n$ payments (number of payments is not $\tilde{\Psi}$).

We will explain this using an example. Consider a life annuity on $(x)$ providing for payments until the death of the annuitant. For the annuities-due, the last payment is made at the beginning of the year of death – that is at time $t = K$, where $K = K(x)$ is curtate lifetime. Then the number of payments $\Psi = K + 1$, and the present value

$$Y = c_0 + c_1 \nu + c_2 \nu^2 + \ldots + c_K \nu^K \quad (0.0.60)$$
In the case of annuity-immediate, the last payment is provided at the same time \( t = K \) since at the end of the year of death the company will not pay. In this case, the number of intervals (or, equivalently, the number of payments) equals \( K \). So, \( \tilde{\psi} = K \) and the present value

\[
\tilde{Y} = c_1 v + c_2 v^2 + \ldots + c_K v^K, \quad \text{if } K > 0 \text{ and } \tilde{Y} = 0 \text{ if } K = 0.
\]  

(0.0.61)

It is obvious that

\[
\tilde{Y} = Y - c_0.
\]  

(0.0.62)

If \( c_0 = c_1 = c_2 = \ldots = c_K = 1 \) then \( a_x = a_x - 1 \).

So, we can write

\[
a_x = a_x - 1 = \frac{1 - A_x}{d} - 1 = \frac{1 - A_x - d}{d} = \frac{1 - A_x - 1 + v}{1 - v} = \frac{v - A_x}{1 - v} = \frac{1 - \frac{A_x}{r}}{1 - \frac{1}{r}} = \frac{1 - \frac{A_x}{r}}{1 - \frac{r - 1}{r}} = \frac{1 - \frac{A_x}{r}}{i} = \frac{1 - rA_x}{i}
\]  

(0.0.63)

where \( r = 1 + i \). So, we have defined annuities-immediate (or payable in arrears).

**3.8.2. Temporary and deferred annuities**

Similarly as in relation (0.0.56) we have

\[
\tilde{a}_{x \mid n} = \frac{1 - A_{x \mid n}}{d},
\]  

(0.0.64)

where \( \tilde{a}_{x \mid n} \) stands for \( n \)-year temporary life annuity, and \( A_{x \mid n} \) is defined by relation (0.0.37).
For annuities payable at the end of each period, according to relation (0.0.57) we have

\[ a_{x+1} = a_{x} - 1 + n E_x = \frac{1 - A_{x+1}}{d} - 1 + n E_x \]  \hspace{1cm} (0.0.65)

where \( n E_x \) is defined by relation (0.0.41).

The actuarial present value of a \( c \)-year deferred life annuity-due with annual payments of 1 is

\[ \ddot{\bar{a}}_x = \ddot{a}_x - \ddot{a}_{x+1} = \frac{1 - A_x}{d} - \frac{1 - A_{x+1}}{d} = \frac{A_{x+1} - A_x}{d} \]  \hspace{1cm} (0.0.66)

In relation (0.0.66) actuarial present value of a \( c \)-year deferred life annuity-due is defined as the difference between the actuarial present value of the whole life annuity-due and the actuarial present value of a \( c \)-year temporary life annuity. If we take relation (0.0.60) as the starting point for defining the actuarial present value of \( c \)-year deferred life annuity-due with annual payments of 1, then

\[ Y = c_1 v + c_2 v^2 + \ldots + c_{c+1} v^{c+1} + \ldots + c_{k} v^{k} \].

It is obvious that the actuarial present value of a \( c \)-year deferred life annuity-due is given by

\[ \ddot{Y} = c_1 v + c_{c+1} v^{c+1} + \ldots + c_{k} v^{k} \].

On the other hand, the connection between payments \( c_1 v + c_2 v^2 + \ldots + c_{c-1} v^{c-1} \) and quantity \( \ddot{a}_{x+1} \) is obvious. Here, quantities \( A_x \) and \( A_{x+1} \) are defined by relations (0.0.16) and (0.0.37), respectively.

Relation (0.0.66) refers to a situation where payments are made at the beginning of each period. The actuarial present value of a \( c \)-year deferred life annuity-immediate with annual payments of 1 is given by the relation
\[ a_x = a_x - a_{x+c} = \frac{1 - A_x - d}{d} - \left( \frac{1 - A_{x+c}}{d} - 1 + c E_x \right) = \]

\[ = \frac{1 - A_{x+c} - A_x - d c E_x}{d}. \]  

Relation (0.0.67) is based on the same approach as relation (0.0.66). Here, \( c E_x \) is defined by relation (0.0.41), as we mentioned earlier.

If we look at the expression \( Y_c = c_1 V + c_{c+1} V^{c+1} + \ldots + c_x V^x \), it is obvious that deferred annuity is actually an annuity that starts at the \( (x+c) \)-th moment with probability \( c p_x \). With \( c p_x \) we have defined the probability that the person of age \( x \) will not die until age \( x+c \). So, we have considered person of age \( x \) (life-age- \( x \) or the symbol \( (x) \)). The future lifetime, or time-until-death after \( x \), is denoted by \( T(x) \). In other words, \( T(x) = x - x \) given \( X > x \). In particular the survival probability is \( P(T(x) > c) = P(X > x+c | X > x) \).

This is the probability that the person of age \( x \) will live at least \( c \) years more, or that the person of age \( x \) will not die for at least \( c \) years more. Given a survival function \( s(x) \),

\[ t p_x = P(X > x + t | X > x) = \frac{P(X > x + t)}{P(X > x)} = \frac{s(x + t)}{s(x)} \]

provided \( s(x + t) \neq 0 \). It is known that that we can view the survival to age \( x \) of particular newborns as a success in \( l_0 \) independent trials (demographers usually set \( l_0 = 100000 \)). Let us denote by \( L(x) \) the number of successes in \( l_0 \) independent trials and \( l_x = E(L(x)) \). According to this, \( L(x) \) has binomial distribution. We can write \( l_x = l_0 s(x) \) and \( s(x) = \frac{l_x}{l_0} \) as well. It is possible to calculate \( c p_x \) as ratio \( l_{x+c} \) and \( l_x \) ratio \( (l_{x+c} \) and \( l_x \) are available in demographic
tables). So, the actuarial present value of a $C$-year deferred life annuity-due with annual payments of 1 can be determined as

$$\ddot{a}_{x:C} = \ddot{a}_x - v^C \cdot p_x \ddot{a}_{x+C}, \quad (0.0.69)$$

and for annuities payable at the end of each period, we have

$$\dot{a}_{x:C} = a_x - v^C \cdot p_x a_{x+C}. \quad (0.0.70)$$

In relations (0.0.69) and (0.0.70) $p_x$ is defined as in relation (0.0.68) or by

$$c \cdot p_x = \frac{l_{x+C}}{l_x}.$$

### 3.8.3. Payments made more frequently than once a year

Consider the case where payments are made $m$ times a year. As we have mentioned earlier, $\bar{i}$ is the effective annual interest rate, and $d = \frac{i}{1+i}$ is the corresponding annual discount rate.

We call $m-th$ the periods of length $\frac{1}{m}$ (if $m=12$, an $m-th$ is a month). So, the actuarial present value of life annuity-due with $m-th$ payments of 1 is

$$\ddot{a}_x^{(m)} = \frac{1}{d^{(m)}} \left(1 - A^{(m)}\right), \quad (0.0.71)$$

where payments are made at the beginning of every $m-th$ period. Here, $\bar{d}^{(m)} = 1 - v^{(m)}$ is the effective discount rate for the $m-th$ period $\left(v^{(m)} = \frac{1}{1+i}\right)$, and quantity $A^{(m)}$ is given by (0.0.22). Relation (0.0.71) is parallel with relation (0.0.56).

If payments are made at the end of every $m-th$ period, then
\( \mathfrak{a}_x^{(m)} = \frac{1}{d^{(m)}} \left( 1 - A^{(m)} \right) - 1 \)  \( (0.0.72) \)

where the same approach is implemented as in relation (0.0.57).

If look at relation (0.0.69), we can define the actuarial present value of temporary annuities with \( m \) – thly payments of 1 as

\[ \mathfrak{a}_{x\mid c}^{(m)} = \mathfrak{a}_x^{(m)} - v^c c \ p_x \mathfrak{a}_{x+c}^{(m)} \]  \( (0.0.73) \)

where \( C \) is the integer. In relation (0.0.73) quantities \( \mathfrak{a}_x^{(m)} \) and \( \mathfrak{a}_{x+c}^{(m)} \) are defined by relation (0.0.71) and probability \( c \ p_x \) is given by relation (0.0.68), and by

\[ c \ p_x = \frac{\mu_{x+c}}{l_x} \]

as well. We can use the same approach for temporary annuities with payments of 1 at the end of every \( m \) – th interval

\[ \mathfrak{a}_{x\mid c}^{(m)} = \mathfrak{a}_x^{(m)} - v^c c \ p_x \mathfrak{a}_{x+c}^{(m)} \]  \( (0.0.74) \)

and here \( C \) is also an integer. In relation (0.0.74) quantities \( \mathfrak{a}_x^{(m)} \) and \( \mathfrak{a}_{x+c}^{(m)} \) are defined by relation (0.0.72) and probability \( c \ p_x \) is given by relation (0.0.68), or

\[ c \ p_x = \frac{\mu_{x+c}}{l_x} . \]

If we look at the relation (0.0.66), then by analogy it is possible to define the actuarial present value of a \( C \) -year deferred life annuity-due with \( m \) – thly payments of 1 as

\[ c \mathfrak{a}_x^{(m)} = \mathfrak{a}_x^{(m)} - \mathfrak{a}_{x\mid c}^{(m)} \]  \( (0.0.75) \)

where quantities \( \mathfrak{a}_x^{(m)} \) and \( \mathfrak{a}_{x\mid c}^{(m)} \) are defined by relation (0.0.71) and (0.0.73) respectively. If we compare relation (0.0.75) with relation (0.0.74), then it is obvious that
We can apply the same logical approach for temporary annuities with payments of 1 at the end of every $m$-th interval. So, we can write

$$c_l a^{(m)}_x = v^c p_x a^{(m)}_{x+c}.$$  \hfill (0.0.76)

4. MODELLING HOME EQUITY CONVERSION LOANS

A homeowner aged 65 or over wants to convert some of his/her home equity into cash without selling or moving out. So $x \geq 65$ and the appraised value of home equity is given by $N$. We have mentioned earlier that if the borrower owns his home with a younger spouse, that spouse also needs to be at least 65 in order to qualify for a home equity conversion loan. The borrower at age 65 and over will receive a certain amount of money. Let us denote that amount of money by $K$. The home equity conversion loan will be repaid with a certain amount of money. This amount is denoted by $K$. Let us define the home equity conversion loan coefficient $\eta$ as

$$\eta \leq \frac{K}{N}.$$  \hfill (0.79)

The home equity conversion loan coefficient should be defined according to the value of the home equity at the present time. The loan will be repaid sometime in the future and the future value of the property is a key factor (Berges, 2004, p. 103). So the home equity conversion loan coefficient should reflect all potential risks that could affect the value of the home equity. Some risks are avoided by strict contract conditions about insurance, home maintenance, etc. But real
estate market volatility is a risk that has to be taken. That risk can be treated as an opportunity.

Let us look at the situation where the borrower receives a lump sum and the loan will be repaid exactly at the moment of death. Then we can write

\[ K_x = \eta N\overline{A}_x \]  

(0.80)

where quantity \( \overline{A}_x \) is defined by (0.10).

When we are considering situations where the borrower receives a lump sum, it makes sense to offer a solution for that situation where the loan is repaid at the end of the year of death. Then we can write

\[ K_x = \eta NA_x, \]  

(0.81)

where quantity \( A_x \) is defined by (0.0.16).

Let us look at the situation where the borrower receives a lump sum and the loan will be repaid at the end of the \( m \)-thly period in which death occurs. Then we can write

\[ K_x = \eta N\overline{A}_x^{(m)} \]  

(0.82)

where quantity \( \overline{A}_x^{(m)} \) is defined by (0.0.22).

We have mentioned earlier that monthly payments can be structured as equal monthly payments for life or equal monthly payments for a fixed period of months.

The borrower will receive a certain amount of money at the beginning of the \( m \)-thly period, or at the end of the \( m \)-thly period. We will denote that amount of money by \( R \). So, if the borrower is going to receive payments at the beginning of the \( m \)-thly period (usually \( m=12 \)) until he dies, then we can write it as follows:
\[ R = \frac{\eta N}{\overline{a}_{x}^{(m)}} \]  

(0.83)

where quantity \( \overline{a}_{x}^{(m)} \) is defined by relation (0.0.71). Let us consider the case where payments are made at the beginning of the year. Then we can write

\[ R = \frac{\eta N}{\overline{a}_{x}} \]

and

\[ R = \frac{\eta N}{\overline{a}_{x}} \]

for payments that are made at the end of the year.

If payments are made at the end of the \( m \)-thly period (usually \( m = 12 \)) until he dies, then we can write it as follows

\[ R = \frac{\eta N}{\overline{a}_{x}^{(m)}} \]  

(0.84)

Now we will define equal monthly payments (or \( m \)-th payments) for a fixed number of months (or fixed number of \( m \)-ths). So, if payments are made at the beginning of the month temporarily for \( c \) years, then we can write it as follows

\[ R = \frac{\eta N}{\overline{a}_{x}^{(m)}} \]  

(0.85)

where quantity \( \overline{a}_{x}^{(m)} \) is defined by relation (0.0.73).

If payments are made at the beginning of the month, temporarily for \( c \) years, then we can write it as follows:
\begin{equation}
R = \frac{\eta N}{a_{xN_1}^{(m)}},
\end{equation}

(0.86)

where quantity $a_{xN_1}^{(m)}$ is defined by relation (0.74) and $m=12$.

There are numerous possibilities for further modelling variations. Here we present basic concepts of home equity conversion loan modelling. The main object of this paper is to provide a perspective on the idea of the home equity conversion loan as a financial product whose pricing is based on life insurance models.

5. CONCLUSION

With a reverse mortgage the borrower can continue to live in his home while he borrows against the value, or equity, of his home. Unlike other types of home equity loans, the borrower does not need to prove that he has enough income to re-pay the loan. In fact, the reverse mortgage loan is not re-paid until the borrower dies or sells his home. Then the lender recovers the amount that the borrower has borrowed. Even with his reverse mortgage, the borrower still retains full ownership of his home and will continue to pay taxes, insurance, and any repairs or maintenance.

Faced with aging populations, countries have to reform both their pension systems, to promote longer working lives, and their labour markets, to ensure that people can actually work longer. The complexity of the issues, the variety of approaches, and the mixed results of adjustments and reforms call for a continuing discussion in an attempt to answer the questions, where do we go from here, and how? Certainly, a home equity conversion loan is not the answer to this problem, but this type of product can be part of a universal answer to the pension system reform problem. A home equity conversion loan can be a useful tool in the process of pension system reform.
Home equity conversion loans are not risk-free for the lender. Some risks can be avoided by strict contract conditions, but real estate market volatility is a risk that has to be taken. That risk can be treated as an opportunity.

REFERENCES


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