ON GAPS BETWEEN BOUNDED OPERATORS

Dragana Cvetković

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Abstract. We consider the distance between an arbitrary bounded operator on a Banach space \( X \) and the null operator. The distance is computed in terms of various gaps. Thus, we generalize the Habibi’s result from [4].

1. Introduction

In [4] Habibi considered the spherical gap between a linear operator and the null operator on finite dimensional Hilbert spaces. We extend (see also [1]) his result to various gaps on arbitrary Banach spaces.

Let \( X, Y \) be arbitrary Banach spaces. We denote by \( \mathcal{G}(X) \), and \( \mathcal{B}(X, Y) \), respectively, the set of all closed subspaces of \( X \) and the set of all bounded linear operators from \( X \) to \( Y \). The closed unit sphere of a Banach space \( X \) is denoted by \( S(X) \). If \( x \) is a vector of a Banach space \( X \) and \( M \) is a subset of \( X \) we put

\[
\text{dist}(x, M) = \inf_{m \in M} \|x - m\|.
\]

If \( M, N \) are subspaces of \( X \), the set of all invertible operators \( C \) on \( X \) such that \( C(M) = N \) is denoted by \( \mathcal{B}(X; M, N)^{-1} \). Let \( X \times Y \) be the space with the norm

\[
\|(x, y)\| = (\|x\|^p + \|y\|^p)^{1/p}, \quad x \in X, \ y \in Y, \ p \geq 1.
\]

Let \( M, N \in \mathcal{G}(X) \). The spherical gap between \( M \) and \( N \) is defined by [2]

\[
\tilde{\Theta}(M, N) = \max \left\{ \tilde{\Theta}_0(M, N), \tilde{\Theta}_0(N, M) \right\},
\]

where

\[
\tilde{\Theta}_0(M, N) = \sup_{m \in S(M)} d(m, S(N)).
\]

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The geometric gap between $M$ and $N$ is defined by [6]

$$\Theta(M, N) = \max \{ \Theta_0(M, N), \Theta_0(N, M) \} ,$$

where

$$\Theta_0(M, N) = \sup_{m \in S(M)} d(m, N).$$

If $A, B \in B(X, Y)$, then their graphs $G(A), G(B)$ are closed subspaces of $X \times Y$, so the gap between the operators $A$ and $B$ is defined as the gap between their graphs, i.e.,

$$\tilde{\Theta}(A, B) = \tilde{\Theta}(G(A), G(B)) \text{ and } \Theta(A, B) = \Theta(G(A), G(B)).$$

Also,

$$\tilde{\Theta}_0(A, B) = \tilde{\Theta}_0(G(A), G(B)) \text{ and } \Theta_0(A, B) = \Theta_0(G(A), G(B)).$$

Obviously,

$$\tilde{\Theta}(A, B) = \max \{ \tilde{\Theta}_0(A, B), \tilde{\Theta}_0(B, A) \} ,$$

$$\Theta(A, B) = \max \{ \Theta_0(A, B), \Theta_0(B, A) \} .$$

In [4] Habibi considered the spherical gap between operators on finite dimensional Hilbert spaces, assuming $p = 2$. We shall consider spherical and geometric gap of bounded operators on Banach spaces, where $p \geq 1$ is arbitrary. Thus, a generalization of Habibi’s result is obtained.

2. Results

First we compute the spherical gap between a bounded operator and the null operator.

**Theorem 2.1.** If $X, Y$ are Banach spaces and $A \in B(X, Y)$, then

$$\tilde{\Theta}(A) = \tilde{\Theta}(A, 0) = \left( \left( 1 - \frac{1}{1 + \| A \|^p} \right)^{1/p} + \frac{\| A \|^p}{1 + \| A \|^p} \right)^{1/p} .$$

**Proof.** Since

$$G(A) = \{ (x, Ax) : x \in X \} \text{ and } G(0) = \{ (x, 0) : x \in X \},$$

it follows that

$$\tilde{\Theta}_0(A, 0) = \sup_{(y, Ay) \in S(G(A))} \inf_{(x, 0) \in S(G(0))} ||(x, 0) - (y, Ay)||$$

$$= \sup_{y \in X} \inf_{\| x \| = 1} \left( || x - y ||^p + || Ay ||^p \right)^{1/p} .$$
Consider the function
\[ f(x, y) = (\|x - y\|^p + \|A y\|)^{1/p}. \]
Suppose that \( x \in X, \ y \in Y \) satisfy
\[ \|x\| = 1, \ \|y\|^p + \|A y\|^p = 1. \]
Then
\[ f(x, y) \geq ((\|x\| - \|y\|)^p + \|A y\|)^{1/p} = ((1 - \|y\|)^p + \|A y\|)^{1/p}. \]
Hence,
\[ \inf_{\|x\| = 1} f(x, y) \leq f\left(\frac{y}{\|y\|}, y\right) = ((1 - \|y\|)^p + \|A y\|)^{1/p}. \]
Also,
\[ \inf_{\|x\| = 1} f(x, y) \leq f\left(\frac{y}{\|y\|}, y\right) = ((1 - \|y\|)^p + \|A y\|)^{1/p}. \]
So, from (2) and (3) it follows that
\[ \inf_{\|x\| = 1} f(x, y) = ((1 - \|y\|)^p + \|A y\|)^{1/p}. \]
According to the condition (1) we have
\[ \|y\| \geq \frac{1}{(1 + \|A\|)^{1/p}}. \]
Since the function \( \phi(t) = (1 - t)^p + 1 - t^p \) is decreasing for \( 0 \leq t \leq 1 \), we have
\[ \tilde{\Theta}_0(A, 0) = \sup_{\|y\|^p + \|A y\|^{1/p} = 1} ((1 - \|y\|^p + 1 - \|A y\|^{1/p})^{1/p} \]
\[ = \left( (1 - \frac{1}{(1 + \|A\|^{1/p})^p} + \frac{|A|^p}{1 + |A|} \right)^{1/p}. \]
Let \( y = c x \), where \( c = \frac{1}{(1 + \|A x\|^{1/p})^{1/p}} \). It is obvious that \( y \) satisfy the condition (1) and \( f(x, y) = \left( (1 - \frac{1}{(1 + \|A x\|^{1/p})^{1/p}} + \frac{|A x|^p}{1 + |A x|} \right)^{1/p}. \)
So,
\[ \inf_{\|y\|^p + \|A y\|^{1/p} = 1} f(x, y) \leq \left( (1 - \frac{1}{(1 + \|A x\|^{1/p})^{1/p}} + \frac{|A x|^p}{1 + |A x|} \right)^{1/p}. \]
The function $\psi(t) = (1 - \frac{1}{(1+t)^{1/p}})^p + \frac{t}{1+t}$ is increasing, so from (6) it follows that

$$\tilde{\Theta}_0(0, A) \leq \sup_{\|x\| = 1} \left( 1 - \frac{1}{(1 + \|Ax\|^p)^{1/p}} \right)^p + \frac{\|Ax\|^p}{1 + \|Ax\|^p}^{1/p}$$

(7)

$$= \left( 1 - \frac{1}{(1 + \|A\|^p)^{1/p}} \right)^p + \frac{\|A\|^p}{1 + \|A\|^p}^{1/p} \quad \Box$$

Finally, from (5) and (7) we have that

$$\tilde{\Theta}(A) = \max \left\{ \tilde{\Theta}_0(A, 0), \tilde{\Theta}_0(0, A) \right\} = \tilde{\Theta}_0(0, A)$$

$$= \left( 1 - \frac{1}{(1 + \|A\|^p)^{1/p}} \right)^p + \frac{\|A\|^p}{1 + \|A\|^p}^{1/p} \quad \Box$$

From the proof of Theorem 2.1 it follows that

$$\tilde{\Theta}_0(0, A) \leq \tilde{\Theta}_0(A, 0) = \left( 1 - \frac{1}{(1 + \|A\|^p)^{1/p}} \right)^p + \frac{\|A\|^p}{1 + \|A\|^p}^{1/p}.$$  

Notice that Habibi’s main result in [4] is a special case of our Theorem 2.1 for $p = 2$.

Now, we consider the geometric gap.

**Theorem 2.2.** If $X, Y$ are Banach spaces and $A \in \mathcal{B}(X, Y)$, then

$$\Theta_0(A, 0) = \frac{\|A\|}{(1 + \|A\|^p)^{1/p}}.$$

**Proof.** It follows that

$$\Theta_0(A, 0) = \sup_{\|y\| = \|Ay\| = 1, \inf_{x \in X} (\|x - y\|^p + \|Ay\|^p)^{1/p}} = \sup_{\|y\| = \|Ay\| = 1} \|Ay\| = \sup_{z \in X} (\|z\|^p + \|Az\|^p)^{1/p} = \frac{\|A\|}{(1 + \|A\|^p)^{1/p}} \quad \Box$$

In the case $p = 2$ we can obtain the following Habibi’s result.

**Theorem 2.3.** If $X$ and $Y$ are Banach spaces, $A \in \mathcal{B}(X, Y)$ and $p = 2$, then the following holds:

$$\Theta(A, 0) = \frac{\|A\|}{(1 + \|A\|^2)^{1/2}},$$
\textbf{Proof.} By Theorem 2.2 it is sufficient to prove
\[ \Theta_0(0, A) \leq \frac{\|A\|}{(1 + \|A\|^2)^{1/2}}. \]

Actually, this is a Habibi's result from [3]. For reader's convenience, we give a complete proof. Consider the function \( f(x, y) = (\|x - y\|^2 + \|Ay\|^2)^{1/2} \). Let \( x \in S(X) \) and \( y = \frac{x}{1 + \|Ax\|^2} \). Then we have
\[ \text{dist}(x, 0), G(A)) \leq f(x, y) = \frac{\|Ax\|}{(1 + \|Ax\|^2)^{1/2}}. \]

Since the function \( \mu(t) = \frac{t}{1 + t^2} \) is increasing, it follows that
\[ \Theta_0(0, A) = \sup_{\|x\| = 1} \text{dist}(x, 0), G(A)) \leq \sup_{\|x\| = 1} \frac{\|Ax\|}{(1 + \|Ax\|^2)^{1/2}} \]
\[ = \frac{\|A\|}{(1 + \|A\|^2)^{1/2}}. \]

Finally, we can prove the following result.

\textbf{Theorem 2.4.} Let \( X, Y \) be Banach spaces and \( X \times Y \) is the space with the norm
\[ \|(x, y)\| = (\|x\|^2 + \|y\|^2)^{1/2}, \quad x \in X, \ y \in Y. \]

If \( A, B \in B(X, Y) \), then
\[ \Theta(A, B) \leq 2(1 + \min\{\|A\|, \|B\|\})^{1/2} \frac{\|A - B\|}{(1 + \|A - B\|^2)^{1/2}}. \]

\textbf{Proof.} From [5], Theorem 2.17, page 204, it follows that
\[ \Theta(A, B) = \Theta((A - B) + B, 0 + B) \leq 2(1 + \|B\|^2)\Theta(A - B, 0), \]
and
\[ \Theta(A, B) = \Theta(0 + A, (B - A) + A) \leq 2(1 + \|A\|^2)\Theta(B - A, 0). \]

Now, by Theorem 2.3 we have that
\[ \Theta(A, B) \leq 2(1 + \|B\|^2) \frac{\|A - B\|}{(1 + \|A - B\|^2)^{1/2}}, \]
and
\[ \Theta(A, B) \leq 2(1 + \|A\|^2) \frac{\|A - B\|}{(1 + \|A - B\|^2)^{1/2}}, \]
i.e.,
\[ \Theta(A, B) \leq 2(1 + \min\{\|A\|, \|B\|\})^{1/2} \frac{\|A - B\|}{(1 + \|A - B\|^2)^{1/2}}. \]

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References


