ON THE CLASSES OF RAPIDLY VARYING FUNCTIONS

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ABSTRACT. The classes $KR_\infty$, $MR_\infty$, $R_\infty$ of rapidly varying functions are natural extensions of Karamata's concept of regular variation. In [2] we introduced a new class $K$ of perfect Karamata's kernels and its subclasses $\Theta$ and $\Sigma$. In this paper we study inclusion properties of these classes and, among other results, we prove $KR_\infty \subset MR_\infty \subset \Sigma \subset \Theta \subset K$.

Introduction

We begin with some definitions from Karamata's theory. A positive measurable function $\ell$ is slowly varying in Karamata's sense if $\ell(\lambda x) \sim \ell(x)$ ($x \to \infty$), for each $\lambda > 0$. Functions of the form $x^\rho \ell(x)$, $\rho \in \mathbb{R}$ are regularly varying with index $\rho$ [1]. For a positive measurable function $f$, define $\tilde{f}$ by $\tilde{f}(x) := \frac{f(x)}{\int_1^x f(t) \ell(t) \, dt}$. It is well known [1], that $\tilde{f}(x) \to \rho$, $0 < \rho < \infty$ ($x \to \infty$), if and only if $f$ is regularly varying function in Karamata's sense with index $\rho$.

From there it follows an extension to the class $\Theta$ of rapidly varying functions. In [2] we gave the following definition.

DEFINITION 1. A positive measurable function $p$ belongs to the class $\Theta$ if and only if $\tilde{p}(x) \to \infty$ ($x \to \infty$).

There is no representation form for the class $\Theta$ since its structure is ambiguous. For example, we showed in [2] that it is not closed under multiplication.

DEFINITION 2. Let $\Sigma$ denote the maximal subclass of $\Theta$ which is closed under multiplication. Then $\Sigma$ consists of all positive measurable functions $s$ such that $s^2 \in \Theta$ [2, Theorem 1].

We also introduced the class $K$ of perfect Karamata's kernels.
**Definition 3.** A positive measurable kernel $C(\cdot)$ belongs to the class $K$ if the asymptotic relation \( \int_1^x f(t)C(t) \, dt \sim f(x) \int_1^x C(t) \, dt \) \( (x \to \infty) \), takes place for every regularly varying function $f(\cdot)$ of arbitrary index.

It is proved in [2] that a necessary and sufficient condition for $C \in K$ is

\[
(1) \quad \int_1^x C(t) \, dt \in \Theta.
\]

Strict inclusion [2],

\[
(2) \quad \Sigma \subset \Theta \subset K,
\]

takes place in the sense that $\Theta/\Sigma$ and $K/\Theta$ are not empty.

From the property of regularly varying function $f$ with index $\rho$, $\forall \lambda > 0$, $f(\lambda x)/f(x) \to \lambda^\rho$ \( (x \to \infty) \), a natural extension to the class $R_\infty$ arises.

**Definition 4.** [1, p. 83] A positive measurable function $f$ belongs to the class $R_\infty$ if $f(\lambda x)/f(x) \to \infty$ \( (x \to \infty) \), for each $\lambda > 1$.

Subclasses of $R_\infty$ are $KR_\infty$ and $MR_\infty$.

**Definition 5.** [1, p. 85] Let $f$ be positive and measurable. Then

(i) $f \in KR_\infty$ if and only if $\liminf_{x \to \infty} \inf_{\lambda \geq 1} \frac{f(\lambda x)}{\lambda^\rho f(x)} = 1$ for every $c \in R$,

(ii) $f \in MR_\infty$ if and only if $\liminf_{x \to \infty} \inf_{\lambda \geq 1} \frac{f(\lambda x)}{\lambda^\rho f(x)} > 0$ for every $d \in R$.

There is strict inclusion [1, p. 83]

\[
(3) \quad KR_\infty \subset MR_\infty \subset R_\infty.
\]

We shall investigate intermediate inclusion properties of the classes $KR_\infty$, $MR_\infty$, $R_\infty$ and $\Sigma$, $\Theta$, $K$ apart from (2) and (3).

**Results**

In all cases there is a strict inclusion property between the classes of rapidly varying functions mentioned above, except in the following one.

**Proposition 1.** The classes $R_\infty$ and $\Theta$ are incomparable i.e., they have not an inclusion property.

Because of the assertion above, there are two inclusion chains. The first one is

**Proposition 2.** An extension of (3) is the following

\[
KR_\infty \subset MR_\infty \subset R_\infty \subset K.
\]

The second one is

**Proposition 3.** An extension of (2) is the following

\[
KR_\infty \subset MR_\infty \subset \Sigma \subset \Theta \subset K.
\]

Therefore the class $K$ includes all known classes of rapidly varying functions in Karamata’s sense.
Proofs

Proof of Proposition 1. In order to prove that the classes $R_\infty$ and $\Theta$ are incomparable, we have to find some positive measurable functions $f$ and $g$ such that $f \in R_\infty$ but $f \notin \Theta$ and $g \in \Theta$ but $g \notin R_\infty$.

An example of $f$ is the next one. Let $f(x) := xe^x$ except at the points $x = e^n$, $n \in N$, where we put $f(e^n) := e^{e^n}$. Now, using Definition 4, it is easy to verify that $f \in R_\infty$. But

$$\tilde{f}(e^n) = e^{e^n-n} \int_1^{e^n} e^t dt \to 0 \quad (n \to \infty).$$

Hence $\lim_{x \to \infty} \tilde{f}(x) = 0$, and $f \notin \Theta$.

An example of $g$ is the following: denote by $(p_n)$, $n \in N$ the sequence of primes and let $g(x) := xe^{x}$ except at the points $x = p_n$ where $g(p_n) := p_ne^{p_n}$. Since $g(x) \geq xe^x$ for $x \geq 1$, we get

$$\tilde{g}(x) \geq xe^x \int_1^x e^t dt \to \infty \quad (x \to \infty);$$

hence $g \in \Theta$. But $\lim_{x \to \infty} \frac{g(2x)}{g(x)} = 2$, i.e., $g \notin R_\infty$.

In order to prove Proposition 2, taking into account (3), we just have to prove that then $f \in K$ whenever $f \in R_\infty$. For this we need the following two lemmas.

Lemma 1. If $f \in R_\infty$, then $\int_1^\infty f(t) dt \in KR_\infty$.

Proof. Denote by $F(x) := \int_1^x f(t) dt$, and let $f \in R_\infty$. Since, for fixed $\lambda > 1$, $f(\lambda t)/f(t) \to \infty$ $(t \to \infty)$ (Definition 4), for any $A > 0$ we can find $t_0$ such that $f(\lambda t) > Af(t)$ for $t > t_0 > 1$. Now, for sufficiently large $x$, we get

$$\frac{F(\lambda x)}{F(x)} = \frac{F(t_0) + \int_{t_0}^x f(t) dt}{F(t_0) + \int_1^x f(t) dt} \geq \frac{F(t_0) + \lambda \int_{t_0}^x f(\lambda t) dt}{F(t_0) + \int_1^x f(t) dt} > \frac{F(t_0) + \lambda A \int_{t_0}^x f(t) dt}{F(t_0) + \int_1^x f(t) dt} > A,$$

since $f(t) \to \infty$ $(t \to \infty)$. Since $A$ can be arbitrary large, we conclude that $F(x) \in R_\infty$. But $F(x)$ is also monotone increasing, hence $[1, p. 85] F \in KR_\infty$. □

Lemma 2. If $g \in MR_\infty$, then $g \in \Theta$. Hence $MR_\infty \subset \Theta$.

This lemma is proved in [1, p. 104].

Proof of Proposition 2. Since $KR_\infty \subset MR_\infty(3)$, from the above lemmas we get $F(x) = \int_1^x f(t) dt \in \Theta$. Applying (1), we obtain $f \in K$. Hence $R_\infty \subset K$.

To prove strict inclusion we shall consider a function $f_1$ defined as: $f_1(x) := e^x$ except at the points $x = 2^n$, $n \in N$ where we put $f_1(2^n) := 2^n$. Then, clearly $\int_1^x f_1(t) dt \in \Theta$; hence by (1), $f_1 \in K$. Yet

$$\lim_{x \to \infty} \frac{f_1(2x)}{f_1(x)} = 2,$$

hence $f_1 \notin R_\infty$. □
Proof of Proposition 3. From (2) and (3) follows that we have to prove that $MR_\infty$ is a proper subclass of $\Sigma$. Applying Lemma 2 we obtain $KR_\infty \subset MR_\infty \subset \Theta$. But from Definition 5 evidently follows that if $f \in MR_\infty$ then also $f^2 \in MR_\infty \subset \Theta$. Hence, according to Definition 2, $MR_\infty \subset \Sigma$.

To prove that the class $MR_\infty$ is a proper subclass of $\Sigma$, we shall consider the following example. Let $f(x) := e^{\log x \exp (\log^2 x)}$, $x \geq 1$ except on intervals of the form $[\exp (n - 1/n), \exp n]$, $n \in \mathbb{N}$, where we put $f(x) := e^{\log x \exp (\log^2 x)/\sqrt{n}}$.

We have to prove that $f \in \Sigma$, i.e., $f^2 \in \Theta$. In order to make calculations simpler, let us change the scale: $x \to \exp x$. In terms of $h(x) := f(e^x)$, we obtain

$$\hat{f}^2(x) = \frac{f^2(x)}{\int_0^x f^2(t)/h(t) \, dt} = \frac{h^2(x)}{\int_0^x h^2(t) \, dt}.$$ 

Then for $x > 0$,

$$\int_0^x h^2(t) \, dt < \int_0^x te^{2x^2} \, dt < e^{2x^2}.$$

Hence for $x \notin \bigcup_{n=1}^{\infty} (n - 1/n, n]$,

$$\hat{f}^2(x) = \frac{h^2(x)}{\int_0^x h^2(t) \, dt} > \frac{xe^{2x^2}}{e^{2x^2}} \to \infty \quad (x \to \infty).$$

If $x \in (n - 1/n, n]$ we obtain

$$\int_0^x h^2(t) \, dt = \int_0^{n-1/n} h^2(t) \, dt + \int_{n-1/n}^x h^2(t) \, dt < \exp(2(n - 1/n)^2) + \frac{e^{2x^2}}{\sqrt{n}}.$$

Hence

$$\hat{f}^2(x) > \frac{xe^{2x^2}/\sqrt{n}}{\exp(2(n - 1/n)^2) + e^{2x^2}/\sqrt{n}} = \frac{x}{1 + \sqrt{n} \exp(2(n - 1/n)^2 - 2x^2)}$$

$$> \frac{n - 1/n}{\sqrt{n} + 1} \to \infty \quad (x \to \infty).$$

Therefore we proved that $f^2 \in \Theta$. By Definition 2 this means that $f \in \Sigma$. Yet

$$\inf_{t \geq 0} \frac{h(n - 1/n + t)}{h(n - 1/n)} = \frac{1}{\sqrt{n}}$$

Hence

$$\liminf_{x \to \infty} \frac{f(\lambda x)}{f(x)} = 0,$$

i.e., by Definition 5(i), $f \notin MR_\infty$. This yields the strict inclusion $MR_\infty \subset \Sigma$. Therefore Proposition 3 is proved.

Remark 1. From Definition 3, it follows that if a function $f$ is in the class $K$, it is still in $K$ if changed in a denumerable number of points.
This remark is e.g., useful if one wants to verify that $\Theta \neq K$. Suppose $f_1 \in K$ is arbitrary. Define $f_0(n) = \int_1^n f_1(s)s^{-1}ds$ for $n = 1, 2, \ldots$ and $f_0 = f_1$ elsewhere. Then $f_0 \notin K, f_0 \notin \Theta$.

A similar remark applies to the proof of Proposition 2. The definition of $f_1 := e^x$ is irrelevant. Take $f_1 \in K$ arbitrary. Then define $f_0(2^n) = 2^n$ for $n \in N$ and $f_0 = f_1$ elsewhere. Then $f_0 \notin R_\infty$ and $F_0 \in K$.

Since there is no representation (except for $KR_\infty$) of rapidly varying functions, any information about it is welcomed. We can provide here such a one.

**Corollary 1.** If $f \in R_\infty$, then

$$\int_1^x f(t)dt = \exp \left( y(x) + z(x) + \int_1^x \frac{u(t)}{t}dt \right),$$

where $y(x)$ is non-decreasing and $z(x) \to 0$, $u(x) \to \infty$ ($x \to \infty$).

This result is a combination of Lemma 1 and well-known representation for the class $KR_\infty$ [1, p. 86].

**References**
