MONOTONE IMAGES OF W-SETS AND HEREDITARILY WEAKLY CONFLUENT IMAGES OF CONTINUA

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Abstract. A proper subcontinuum $H$ of a continuum $X$ is said to be a $W$-set provided for each continuous surjective function $f$ from a continuum $Y$ onto $X$, there exists a subcontinuum $C$ of $Y$ that maps entirely onto $H$. Hereditarily weakly confluent (HWC) mappings are those with the property that each restriction to a subcontinuum of the domain is weakly confluent. In this paper, we show that the monotone image of a $W$-set is a $W$-set and that there exists a continuum which is not in class $W$ but which is the HWC image of a class $W$ continuum.

1. Introduction

In what follows, a continuum is a compact, connected metric space, and the term map is used to denote a continuous function. It is known that monotone images of class $W$ continua are in class $W$, as shown in [1]. In the summer of 2000, two questions arose related to this result. First, in personal communication, W. J. Charatonik asked whether HWC maps preserve membership in class $W$. We answer his question in the negative in Section 3. Second, while discussing approaching continuum theory from an analytical viewpoint and attempting to characterize continua which are not intrinsic $W$-sets, the idea of examining the preimages of such continua under certain types of maps arose. We give a related theorem in Section 4.

2. Definitions

A proper subcontinuum $H$ of a continuum $X$ is said to be a $W$-set provided for each continuous surjective function $f$ from a continuum $Y$ onto $X$, there exists

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a subcontinuum $C$ of $Y$ that maps entirely onto $H$. A continuum each proper subcontinuum of which is a $W$-set is said to be in class $W$. A weakly confluent map $f : X \to Y$ is one so that, for each subcontinuum $K$ of the range, there is at least one component $C$ of $f^{-1}(K)$ so that $f(C) = K$. A hereditarily weakly confluent (HWC) map $f : X \to Y$ is one so that for each subcontinuum $K$ of $X$, $f|_K$ is weakly confluent. A monotone map $f : X \to Y$ is one so that $f^{-1}(y)$ is connected for each $y \in Y$. The Hausdorff distance between two compact sets $A$ and $B$ is defined to be

$$d_H(A, B) = \inf \{ \epsilon > 0 \mid A \subseteq B_\epsilon(B) \text{ and } B \subseteq B_\epsilon(A) \}.$$ 

3. HWC maps and class W

W. J. Charatonik asked whether the HWC image of a class $W$ continuum is necessarily in class $W$. The answer is no.

**Theorem 1.** There exist a a continuum $M$ in class $W$, a continuum $X$, and a surjective HWC map $f : M \to X$ where $X$ is a continuum not in class $W$.

**Proof.** First, consider the continuum formed when one takes a two endpoint Knaster-type continuum, which can be realized as an inverse limit on arcs with a three-pass bonding map, and joins the two endpoints. The result, which we will denote by $X$, is an indecomposable continuum homeomorphic to the one illustrated on the right in Figure 1. It is clear that $X$ is not in class $W$, since for the quotient map itself, there is no continuum in the domain which is mapped onto the arc from $f(p')$ to $f(q')$.

Denote by $C$ the composant of $X$ containing $f(p)$, the joining point. In the strip $\mathbb{R} \times [0,1]$, consider the collection of straight line segments of the following form: $C_0$ is the straight line segment from $(0,1)$ to $(-1,1/2)$. Then, for each positive integer $n$, let $C_n$ be the straight line segment from $((-1)n \cdot n, 1/(n+1))$ to $((-1)n+1 \cdot (n+1), 1/(n+2))$. Let $\hat{Y} = \bigcup_{n \geq 0} C_n$, and observe that $\hat{Y}$ is a connected set containing all of $\mathbb{R} \times \{0\}$ in its closure.
To construct $M$, which will be a subset of $X \times [0, 1]$, first consider the straight line $\ell$ in the plane connecting the points $a$ and $b$. There is a natural surjective and injective map $\hat{g}$ from $\mathbb{R}$ to $C$ with the following properties: first, that $\hat{g}(0) = f(p)$ and second, that for each integer $n$ in $\mathbb{R} \setminus \{0\}$, $\hat{g}(n) \in \ell$. Extend $\hat{g}$ to $g : \mathbb{R} \times [0, 1] \to C \times [0, 1]$ by setting $g(x, t) = (\hat{g}(x), t)$. Let $Y = g(Y)$ and define $M = X \cup Y$.

Observe that $Y$ contains $C$, and since $C$ is dense in $X$, $Y = M$. It is easily verified that $M$ is in class $W$ (for example by using Theorem 67.1 of [3] and Proposition 4 of [4]). Let $\pi : X \times [0, 1] \to X$ be simple projection map, and we will now show that $\pi|_M$ is HWC. Let $K$ be any subcontinuum of $M$. If $K \subseteq X$ or $X \subseteq K$, then since $\pi|_X$ is essentially the identity, $\pi|_K$ is clearly weakly confluent. If $K \not\subseteq X$ and $X \not\subseteq K$, then $K \subseteq Y$, since $X$ is a $C$-set in $M$. For $K \subseteq Y$, $K$ is an arc, and since arcs are in class $W$, $\pi|_K$ is weakly confluent. Hence $\pi|_M$ is HWC. Thus $X$ is the HWC image of a class $W$ continuum, and $X$ is not in class $W$. □

4. Monotone maps and $W$-sets

In investigating what properties of a subcontinuum imply that it is not an intrinsic $W$-set, the idea arose that perhaps considering the preimages of such a subcontinuum under various types of maps might be informative. One of the questions that was generated by that discussion was about the monotone preimage of a continuum which was not an intrinsic $W$-set. After answering this question, we realized that there was a different statement of the same result which might be more useful.

The proof of our Theorem 2 depends on the following theorem that will appear in [2]

**Theorem.** [2, Theorem 7] A subcontinuum $H$ of a continuum $X$ is a $W$-set in $X$ if and only if for each $\epsilon > 0$ there is a pair $H_1, H_2$ of compact subsets of $X$ so that any continuum $C$ intersecting both $H_1$ and $H_2$ which is not separated by $H_1 \cup H_2$ has Hausdorff distance from $H$ less than $\epsilon$.

**Theorem 2.** Let $X$ be a continuum with $W$-set $H$, and let $f : X \to Y$ be a map of $X$ to a continuum $Y$. If $f$ is monotone and $f(H)$ a proper subcontinuum of $f(X)$, then $f(H)$ is a $W$-set in $f(X)$.

**Proof.** Let $H$ be a $W$-set in continuum $X$, and let $f : X \to Y$ be a monotone map so that $f(H)$ is nondegenerate. Without loss of generality, assume that $f$ is surjective. Given any $\epsilon > 0$ so that $\epsilon < \frac{1}{4}(\text{diam}(f(H)))$, there is a $\delta > 0$ so that if $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$, then $d(f(x_1), f(x_2)) < \epsilon$. Since $H$ is a $W$-set, there exist two compact subsets, $H_1$ and $H_2$, so that for any continuum $C$ from $H_1$ to $H_2$ which is not separated by $H_1 \cup H_2$, $d_H(C, H) < \delta$.

Assume, for the sake of contradiction, that there is a point $p \in f(H_1) \cap f(H_2)$. Then $f^{-1}(p)$ is a continuum intersecting both $H_1$ and $H_2$, $f^{-1}(p)$ must therefore contain a continuum $C$ irreducible between $H_1$ and $H_2$, which by thus must have $d_H(C, H) < \delta$. Therefore $d_H(f(p), f(H)) < \epsilon$, which implies that $f(H) \subset B_\epsilon(p)$. This contradicts our choice of $p$, so $f(H_1)$ and $f(H_2)$ must be disjoint.
If \( C \) is a continuum from \( f(H_1) \) to \( f(H_2) \) not separated by their union, then \( f^{-1}(C) \) is a continuum intersecting \( H_1 \) and \( H_2 \). Define \( M \) and \( N \) as follows:

\[
M = f^{-1}\left(C \setminus (f(H_1) \cup f(H_2))\right), \quad N = f^{-1}\left(C \setminus (f(H_1) \cup f(H_2))\right).
\]

Observe that \( N \) is a subcontinuum of \( M \). Let \( P \) and \( Q \) be subcontinua of \( M \) so that \( P \) is irreducible between \( H_1 \cap M \) and \( N \) and \( Q \) is irreducible between \( H_2 \cap M \) and \( N \). The continuum \( N \cup P \cup Q \) intersects \( H_1 \) and \( H_2 \) but is not separated by their union, so \( d_H(P \cup N \cup Q, H) < \delta \). Hence each point of \( f(H) \) is within \( \epsilon \) of \( f(P \cup N \cup Q) \subseteq C \), and each point of \( C \) is either in \( f(P \cup N \cup Q) \), and hence within \( \epsilon \) of \( f(H) \), or in \( f(H_1) \cup f(H_2) \), which must be within \( \epsilon \) of \( F(H) \), since \( H_1 \) and \( H_2 \) must both be within \( \delta \) of \( H \). Thus \( d_H(C, H) < \epsilon \).

This satisfies the conditions in Theorem [2, Theorem 7] for \( f(H) \) to be a \( W \)-set in \( f(X) \).

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