MÖBIUS TRANSFORMATIONS
AND MULTIPLICATIVE REPRESENTATIONS
FOR SPHERICAL POTENTIALS

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Abstract. For the unit spheres $S^n \subset \mathbb{R}^{n+1}$ and $S^{2n-1} \subset \mathbb{R}^{2n} = \mathbb{C}^n$ we prove the following identities for two classical potentials

$$
\int_{S^n} \frac{f(y)}{|x-y|^{n+\alpha}} d\sigma_y = \frac{1}{|1-|x||^{n+\alpha}} \int_{S^n} \frac{f(T_n(x)(y))}{|x-y|^{n+\alpha}} d\sigma_y,
$$

$$
\int_{S^{2n-1}} \frac{F(\zeta) d\sigma_\zeta}{|1-|\zeta||^{n+\alpha}} = \int_{S^{2n-1}} \frac{F(\Phi_n(x)(\zeta)) d\sigma_\zeta}{|1-|z, \zeta||^{n+\alpha}},
$$

where $x \in \mathbb{R}^{n+1}$, $|x| \neq 0$ and $|x| \neq 1$, $z \in \mathbb{C}^n$, $|z| < 1$. $T_n(x)$ and $\Phi_n(x)$ are explicit involutions of $S^n$ and $S^{2n-1}$ respectively. Some applications of these formulas are also considered.

1. Introduction

The aim of this paper is to present a new approach to study boundary behavior of classical potentials using Möbius transformations in two and several dimensions. We consider two spherical potentials in the spaces $\mathbb{R}^{n+1}$ and $\mathbb{C}^n$ for $n \geq 1$. The first one is the Riesz potential

$$
P_{n, \alpha}(x, f) = \int_{S^n} \frac{f(y)}{|x-y|^{n+\alpha}} d\sigma_y
$$

of the sphere $S^n = \{y \in \mathbb{R}^{n+1} : |y| = 1\}$ in $\mathbb{R}^{n+1}$ for $|x| \neq 1$, and the second is the complex potential

$$
Q_{n, \alpha}(z, F) = \int_{S^{2n-1}} \frac{F(\zeta)}{|1-|z, \zeta||^{n+\alpha}} d\sigma_\zeta
$$

of the sphere $S^{2n-1} = \{\zeta \in \mathbb{C}^n : |\zeta| = 1\}$ in $\mathbb{C}^n$ for $|z| < 1$. In (1) and (2) $d\sigma_y$ and $d\sigma_\zeta$ denote the differential elements of surface area of the spheres $S^n \subset \mathbb{R}^{n+1}$ and $S^{2n-1} \subset \mathbb{R}^{2n}$, respectively, and $(z, \zeta) = z_1 \zeta_1 + z_2 \zeta_2 + \cdots + z_n \zeta_n$ is the scalar product in $\mathbb{C}^n$.

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It is very well known that the classical methods use some special additive representations of (1) and (2) to study their boundary behavior (see, for instance, \[1, 3, 5, 6\]). We will give new formulas to find the singularities of spherical potentials in the case, when \(\alpha\) is a complex number such that \(\Re\alpha > 0\).

Namely, for (1) and (2) we obtain multiplicative representations which explicitly give the principal singularities of these potentials near the spheres \(S^n\) and \(S^{2n-1}\) respectively. Moreover, we apply the multiplicative representations to find sharp estimates for the functions

\[
|1 - |x|^2|^{\beta} P_{n,\alpha}(x, f) \quad \text{and} \quad |1 - |z|^2|^{\beta} Q_{n,\alpha}(z, F)
\]

when \(\beta \geq \Re\alpha\) and the densities \(f\) and \(F\) belong to \(L^q\) with \(q > 1\). We also show that the multiplicative representations may be used to prove Fatou type theorems.

The paper is organized as follows. In Section 2 the Riesz potential \(P_{n,\alpha}\) is considered. In Section 3 we study the complex potential (2) in some details. It is clear that \(P_{1,\alpha}(x, f) \equiv Q_{1,\alpha}(z, F)\) for \(f = F\) and \(x := (x_1, x_2), z := (x_1 + i x_2)\), but \(Q_{n,\alpha}(z, F)\) does not reduce to \(P_{2n-1,\alpha}(x, f)\) for \(n \geq 2\).

2. Riesz spherical potentials

We intend to transform integral (1) by a change of variables using Möbius transformations. Consider first the trivial case \(n = 0\). We can take \(S^0 = \{-1, 1\}\) and

\[
P_{0,\alpha}(x, f) := \frac{f(-1)}{|x + 1|^\alpha} + \frac{f(1)}{|x - 1|^\alpha}, \quad x \in \mathbb{R} \setminus S^0,
\]

for any function \(f : S^0 \to \mathbb{C}\). If \(T_0 : S^0 \to S^0\) is involute, i.e., \(T_0(1) = -1, T_0(-1) = 1\), then the following identity

\[
P_{0,\alpha}(x, f) = \frac{|x - 1|^\alpha f(-1) + |x + 1|^\alpha f(1)}{|1 - x^2|^\alpha} = \frac{1}{|1 - x^2|^\alpha} P_{0,\alpha}(x, f \circ T_0)
\]

is valid in \(\mathbb{R} \setminus S^0\). Surprisingly, this elementary formula has a direct extension to the case \(n \geq 1\).

For \(n \geq 1\) and every fixed \(x \in \mathbb{R}^{n+1} \setminus S^n, |x| \neq 0\), we will consider the following Möbius transformations of \(\mathbb{R}^{n+1}\n
\[
T_{n,x}(y) = \begin{cases} 
    x + \frac{(|x|^2 - 1)(y - x)}{|y - x|^2}, & \text{if } |x| > 1, \\
    x - \frac{(|x|^2 - 1)(y - x)}{|y - x|^2}, & \text{if } 0 < |x| < 1.
\end{cases}
\]

For fixed \(x\) the transformation \(T_{n,x}\) is a conformal automorphism of the unit ball \(B_{n+1} := \{y \in \mathbb{R}^{n+1} : |y| \leq 1\}\) (see [1]) and the restriction \(T_{n,x} \mid S^n\) presents the standard inversion of \(S^n\) about the sphere \(S^{n-1} = \{y \in S^n : |y - x| = \sqrt{1 - |x|^2}\}\).

**Theorem 1.** Suppose that \(n \geq 1\) and \(f \in L^1(S^n)\). For any \(\alpha \in \mathbb{C}\) and for all \(x \in \mathbb{R}^{n+1} \setminus S^n, |x| \neq 0\), the following identity is valid

\[
\int_{S^n} \frac{f(y)}{|x - y|^{n+\alpha}} d\sigma_y = \frac{1}{|1 - |x|^2|^\alpha} \int_{S^n} \frac{f(T_{n,x}(y))}{|x - y|^{n-\alpha}} d\sigma_y
\]
Proof. Let \( x \in \mathbb{R}^{n+1} \setminus S^n, |x| \neq 0 \). To simplify computations it is convenient to use a new orthonormal basis \((e_1, e_2, \ldots, e_{n+1})\) obtained by a rotation of \( \mathbb{R}^{n+1} \) about the origin and such that \( x = |x|e_1 \).

Suppose that
\[
y = \sum_{k=1}^{n+1} y_k e_k \quad \text{and} \quad u = T_{n,x}(y) = \sum_{k=1}^{n+1} u_k e_k.
\]

Straightforward computations using (3) give
\[
(5) \quad u_1 = T_{1,|x|}(y_1) := \frac{2|x| - (1 + |x|^2)y_1}{1 + |x|^2 - 2|y_1|y_1}
\]
and
\[
(6) \quad u_k = \frac{|1 - |x|^2|}{1 + |x|^2 - 2|y_1|y_1} y_k = \sqrt{\frac{1 - u_1^2}{1 - y_1^2}} y_k, \quad 2 \leq k \leq n + 1,
\]
in both cases: \( |x| > 1 \) or \( 0 < |x| < 1 \). To deduce the second equalities for \( u_k \) in (6) we used the following consequence of (5):
\[
(7) \quad 1 - u_1^2 = \frac{(1 - |x|^2)^2}{(1 + |x|^2 - 2|y_1|y_1)^2}(1 - y_1^2).
\]
Moreover, equality (5) implies that \( y_1 = T_{1,|x|}(u_1) \), hence
\[
(8) \quad 1 - y_1^2 = \frac{(1 - |x|^2)^2}{(1 + |x|^2 - 2|y_1|y_1)^2}(1 - u_1^2).
\]
Using (5) and (6) we also obtain that \( u = T_{n,x}(y) \in S^n \) for any \( y \in S^n \) and \( T_{n,x} | S^n \) is an involution of \( S^n \).

From (7) and (8) it follows that
\[
(1 + |x|^2 - 2|y_1|)(1 + |x|^2 - 2|u_1|) = (1 - |x|^2)^2
\]
which is equivalent to the equality
\[
(9) \quad |x - u| \cdot |x - y| = |1 - |x|^2|
\]
for any \( y \in S^n \) and \( u = T_{n,x}(y) \).

Thus,
\[
(10) \quad \int_{S^n} \frac{f(u)}{|x - u|^{n+\alpha}} d\sigma_u = \frac{1}{|1 - |x|^2|^{n+\alpha}} \int_{S^n} f(T_{n,x}(y)) |x - y|^{n+\alpha} I(y) d\sigma_y,
\]
where \( I(y) = d\sigma_u/d\sigma_y \) \( (u = T_{n,x}(y)) \) is the Jacobian of the map \( T_{n,x} | S^n \). To compute \( I(y) \) we consider a diffeomorphism \( K : B_{n+1} \to B_{n+1} \) defined by
\[
(K | S^n)(\xi) = (T_{n,x} | S^n)(\xi) \quad \text{for} \quad \xi \in S^n
\]
and
\[
v = K(\xi) = \sum_{k=1}^{n+1} v_k e_k \quad \text{for} \quad |\xi| < 1,
\]
where
\begin{equation}
(11) \quad v_1 = T_{1,|x|}(\xi_1), \quad v_k = \sqrt{\frac{1 - v_1^2}{1 - \xi_k^2}} \text{ for } 2 \leq k \leq n + 1.
\end{equation}

For any $\xi \in S^n$ and $v = K(\xi)$ one has
\[
I(y) = \lim_{\xi \to y} \left( \frac{1 - |\xi|^2}{|v|^2} \right)^{1+\frac{n}{2}} \left[ \frac{1 - |\xi|^2}{|v|^2} \right]^{1+\frac{n}{2}}.
\]

Since
\[
\frac{\partial v_1}{\partial \xi_1} = \frac{1 - v_1^2}{1 - \xi_1^2}, \quad \frac{\partial v_k}{\partial \xi_k} = \sqrt{\frac{1 - v_1^2}{1 - \xi_k^2}} \text{ for } k \geq 2
\]
and
\[
\frac{\partial v_k}{\partial \xi_j} = 0 \text{ for } k \geq 1 \text{ and } j > k,
\]
we have
\[
I(y) = \lim_{\xi \to y} \left( \frac{1 - |\xi|^2}{|v|^2} \right)^{1+\frac{n}{2}}.
\]

From (11) it follows that
\[
1 - |v|^2 = \frac{1 - v_1^2}{1 - \xi_1^2}(1 - |\xi|^2).
\]

Using this and the formula (8) for $v = T_{n,x}(y) = K(y) \in S^n$ we obtain
\begin{equation}
(12) \quad I(y) = \left( \frac{1 - |x|^2}{|x - y|^{2n}} \right)^{1+\frac{n}{2}}, \quad y \in S^n.
\end{equation}

Formulas (10) and (12) imply (4). Thus, the proof of Theorem 1 is complete. 

**Corollary 1.1.** Let $F \in L^q(S^n), q > 1$. If $\beta = \Re \alpha + n/q > 0$ then for any fixed $x \in \mathbb{R}^{n+1} \setminus S^n$

\begin{equation}
(13) \quad \sup_{\|f\| = 1} \left( \int_{S^n} \frac{|1 - |x|^2|^\beta f(y)}{|x - y|^{n+\alpha}} \, d\sigma_y \right)^{1/q} = \left( \int_{S^n} \frac{d\sigma_y}{|x - y|^{n-\beta t}} \right)^{1/q},
\end{equation}

where $t = (q - 1)/q < 1$ and
\[
\|f\|_q = \left( \int_{S^n} |f(y)|^q \, d\sigma_y \right)^{1/q}.
\]

**Proof.** According to Hölder’s inequality
\begin{equation}
(14) \quad \sup_{\|f\| = 1} \|P_{n,\alpha}(x, f)\| = P_{n,\beta}^{1/q}(x, 1).
\end{equation}

Applying Theorem 1 we obtain
\begin{equation}
(15) \quad P_{n,\beta}(x, 1) = \frac{1}{|1 - |x|^2|^\beta} P_{n,\alpha}(x, 1).
\end{equation}

Equalities (14) and (15) imply (13). 

\qed
By virtue of well-known properties of Riesz potentials the integral \( P_{n, \beta}(x, 1) \) depends on \(|x|\) only and has three critical points that are \(|x| = 0\), \(|x| = 1\) and \(|x| = \infty\). Compare \( P_{n, \beta}(0, 1) \), \( P_{n, \beta}(1, 1) \) and \( P_{n, \beta}(\infty, 1) \) one may compute its maximum and minimum for \( 0 \leq |x| \leq 1 \) or \( 1 \leq |x| \leq \infty\). In particular, if \( n \geq 2\), \( 0 \leq n - \beta t \leq n - 1 \), then \( P_{n, \beta}(0, 1) \geq \max\{P_{n, \beta}(1, 1), P_{n, \beta}(\infty, 1)\} \).

Consequently, (13) implies the sharp estimate

\[
|1 - |x|^2|^{\beta}|P_{n, \alpha}(x, f)| \leq \sigma_n^{1/\beta} ||f||_q, \quad \forall x \in \mathbb{R}^{n+1} \setminus S^n,
\]

where \( \sigma_n = \frac{2\pi^{(n+1)/2}}{\Gamma((n + 1)/2)} \) is “the surface area” of \( S^n \) in \( \mathbb{R}^{n+1} \). Equality in (16) occurs for \(|x| = 0\) and \( f(y) \equiv \text{const} \).

Using classical methods for Poisson’s integrals (see, for instance, [1]) one may prove the following Fatou’s theorem for \( P_{n, \alpha}(x, f) \) in the case \( \Re \alpha > 0 \) and \( f \in L^1(S^n) \) : for almost all \( \xi \in S^n \)

\[
\lim_{|x| \to |\xi|} |1 - |x|^2|^{\alpha}P_{n, \alpha}(x, f) = 2^{\alpha - n/2} \frac{\Gamma((\alpha)/2)}{\Gamma((\alpha + n)/2)} f(\xi),
\]

(17) where \( M \) is a positive constant.

In the next Corollary 1.2 we examine (17) for a particular case when (17) is a simple consequence of Theorem 1 and a property of \( T_{n, x} \).

**Corollary 1.2.** If \( \Re \alpha > 0 \), \( f \in L^\infty(S^n) \) and \( f \) is continuous at the point \( \xi \in S^n \), then

\[
\lim_{|x| \to |\xi|} |1 - |x|^2|^{\alpha}P_{n, \alpha}(x, f) = 2^{\alpha - n/2} \frac{\Gamma((\alpha)/2)}{\Gamma((\alpha + n)/2)} f(\xi).
\]

**Proof.** According to Theorem 1 we have to prove that

\[
\lim_{|y| \to |\xi|} \int_{S^n} \frac{f(T_{n, x}(y))}{|x - y|^{n-\alpha}} d\sigma_y = f(\xi) \int_{S^n} \frac{d\sigma_y}{|\xi - y|^{n-\alpha}} = \frac{\Gamma((\alpha)/2)}{\Gamma((\alpha + n)/2)} f(\xi),
\]

which is equivalent to

\[
A(x, \xi) = \int_{S^n} \frac{f(T_{n, x}(y))}{|x - y|^{n-\alpha}} d\sigma_y \to 0 \quad \text{as} \ x \to \xi, \ |x| \neq 1.
\]

Since Hölder’s inequality on can write

\[
|A(x, \xi)| \leq C \left( \int_{S^n} |f(T_{n, x}(y)) - f(\xi)|^q d\sigma_y \right)^{1/q},
\]

where \( C \) is a constant.

From (3) it follows that

\[
\lim_{|x| \to |\xi|} T_{n, x}(y) = \xi, \quad \forall y \in S^n \setminus \{\xi\}.
\]

Consequently, \( f(T_{n, x}(y)) \to f(\xi) \) as \( x \to \xi \), \( |x| \neq 1 \) for any \( y \in S^n \setminus \{\xi\} \) and \( \|f \circ T_{n, x} - f(\xi)\| \to 0 \) as \( x \to \xi \), \( |x| \neq 1 \) by Lebesgue’s theorem on the majorized convergence. This completes the proof of Corollary 1.2. \( \square \)
The function $P_{1,\alpha}(r, 1)$ is used in many problems related to the spaces of functions analytic in the unit disk. We add to known results (see [2], [4]) the following assertion. We will need the beta function
\[
B\left(\frac{1}{2}, \frac{\alpha}{2}\right) = \frac{\sqrt{\pi} \Gamma(\alpha/2)}{\Gamma((\alpha + 1)/2)}.
\]

**Corollary 1.3.** If $0 \leq r < 1$, $\alpha > 0$ and $\alpha \neq 1$ then
\[
\frac{2\pi}{(1 - r^2)^\alpha} \leq \int_0^{2\pi} \frac{d\theta}{|1 - re^{i\theta}|^{1+\alpha}} < \frac{2\alpha B(1/2, \alpha/2)}{(1 - r^2)^\alpha}.
\]

Equality in the left-hand side inequality occurs if and only if $r = 0$. The right-hand side inequality is sharp asymptotically as $r \to 1 - 0$.

**Proof.** By Theorem 1
\[
\int_0^{2\pi} \frac{d\theta}{|1 - re^{i\theta}|^{1+\alpha}} = \frac{1}{(1 - r^2)^\alpha} \int_0^{2\pi} \frac{d\theta}{|1 - re^{i\theta}|^{1-\alpha}}.
\]

According to Hardy’s theorem $P_{1,\alpha}(., 1)$ is an increasing function in $[0, 1]$ if $\alpha \neq 1$. Consequently, for any $r \in (0, 1)$, $\alpha > 0$ and $\alpha \neq 1$
\[
(1 - r^2)^\alpha P_{1,\alpha}(r, 1) > P_{1,\alpha}(0, 1) = 2\pi,
\]
\[
(1 - r^2)^\alpha P_{1,\alpha}(r, 1) < \lim_{r \to 1^-} (1 - r^2)^\alpha P_{1,\alpha}(r, 1) = P_{1,\alpha}(1, 1) = 2\alpha B(1/2, \alpha/2).
\]

Two last formulas complete the proof of Corollary 1.3. \(\square\)

### 3. The Potential of $S^{2n-1}$ in $\mathbb{C}^n$

Let $B$ be the unit ball $\{\zeta \in \mathbb{C}^n : |\zeta| < 1\}$, $\partial B = S^{2n-1}$. For fixed $z \in B \setminus \{0\}$ we consider the biholomorphic map $\Phi_{n, z}$ of $B$ onto $B$ defined as follows (see [5]):
\[
\Phi_{n, z}(\zeta) = \frac{z - p_z(\zeta) - \sqrt{1 - |z|^2}(\zeta - p_z(\zeta))}{1 - (\zeta, z)}, |\zeta| \leq 1,
\]
where
\[
p_z(\zeta) = \frac{z}{|z|^2}(\zeta, z).
\]

It is known (see [5]) that
(i) $\Phi_{n, z}$ is an involution, i.e., $\Phi_{n, z}(\Phi_{n, z}(\zeta)) = \zeta$ for any $\zeta \in \partial B$;
(ii) $\Phi_{n, z}$ satisfies the conditions
\[
\Phi_{n, z}(z) = 0, \Phi_{n, z}(z/|z|) = -z/|z|, \Phi_{n, z}(\zeta) \in S^{2n-1} \quad \text{and} \quad \Phi_{n, z}(\zeta) \neq \zeta \quad \text{for any} \quad \zeta \in S^{2n-1};
\]
(iii) $\Phi_{n, z}| S^{2n-1} : S^{2n-1} \to S^{2n-1}$ is a diffeomorphism;
(iv) there is the identity
\[
1 - (\Phi_{n, z}(\zeta), \Phi_{n, z}(w)) = \frac{(1 - |z|^2)(1 - (\zeta, w))}{(1 - (\zeta, z))(1 - (z, w))}.
\]

For $Q_{n, \alpha}(z, F)$ we have the following analog of Theorem 1. Note that the assertion of Theorem 2 is known in the case $\alpha = n$ (see [5, Chapter 1]).
**Theorem 2.** Suppose that $\alpha \in \mathbb{C}$, $F \in L^1(S^{2n-1})$. For any $z \in B \setminus \{0\}$ the following identity is valid:

\[
\int_{S^{2n-1}} \frac{F(\zeta) d\sigma_\zeta}{|1 - (z, \zeta)|^{n+\alpha}} = \frac{1}{(1 - |z|^2)^\alpha} \int_{S^{2n-1}} \frac{F(\Phi_n z(\zeta)) d\sigma_\zeta}{|1 - (z, \Phi_n z(\zeta))|^{n+\alpha}},
\]

where $S^{2n-1} = \partial B = \{ \zeta \in \mathbb{C}^n : |\zeta| = 1 \}$.

**Proof.** Let $z \in B \setminus \{0\}$. Taking $w = \Phi_n z(\zeta)$, $\zeta \in S^{2n-1}$, we have

\[
\int_{S^{2n-1}} \frac{F(\zeta) d\sigma_\zeta}{|1 - (z, \zeta)|^{n+\alpha}} = \int_{S^{2n-1}} \frac{F(\Phi_n z(\zeta)) d\sigma_\zeta}{|1 - (z, \Phi_n z(\zeta))|^{n+\alpha}}.
\]

From the properties (i), (ii) and (iv) we have $\zeta = \Phi_n z(w)$ and

\[
1 - (w, \zeta) = \frac{(1 - |z|^2)(1 - (\zeta, w))}{(1 - (\zeta, z))(1 - (z, w))}, \quad (\zeta, w) \neq 1.
\]

Consequently, for any $\zeta \in S^{2n-1}$ and $w = \Phi_n z(\zeta)$

\[
|1 - (z, w)| \cdot |1 - (z, \zeta)| = 1 - |z|^2
\]

According to Theorem 3.3.8 in [5] the Jacobian

\[
I(\zeta) = \frac{d\sigma_w}{d\sigma_\zeta} = \frac{(1 - |z|^2)^n}{|1 - (z, \zeta)|^{2n}}.
\]

From (19), (20) and (21) we have (18) immediately. The proof of Theorem 2 is complete.\[\square\]

In [5, Proposition 1.4.10], for $-\frac{n + \alpha}{2} \notin \mathbb{N}$ it is proved that

\[
Q_{n, \alpha}(z, 1) = \frac{\sigma_{2n-1} \Gamma(n)}{\Gamma^2((n + \alpha)/2)} \sum_{k=0}^{\infty} \frac{\Gamma^2(k + (n + \alpha)/2)}{\Gamma(k + 1) \Gamma(k + n)} |z|^{2k}
\]

and that

\[
Q_{n, \alpha}(z, 1) \approx (1 - |z|^2)^{-\alpha} \quad \text{for} \quad \alpha > 0.
\]

It is to note that

\[
\sigma_{2n-1} = \int_{S^{2n-1}} d\sigma_\zeta = \frac{2\pi^n}{\Gamma(n)}
\]

is "the surface area" of $S^{2n-1}$ in $\mathbb{R}^{2n}$, and in [5] the normalized measure

\[
d\sigma(\zeta) = d\sigma_\zeta /\sigma_{2n-1}
\]

is considered. Hence, $Q_{n, \alpha}(z, 1)/\sigma_{2n-1}$ is $I_c(z)$ from [5, Chapter 1], with $c = \alpha$.

Using Theorem 2 and the series (22) we get a refined version of (23).

**Corollary 2.1.** If $\alpha > 0$, $z \in B_n$ and $F \in L^\infty(S^{2n-1})$ then

\[
\left| \int_{S^{2n-1}} \frac{F(\zeta) d\sigma_\zeta}{|1 - (z, \zeta)|^{n+\alpha}} \right| \leq \frac{2\pi^n \Gamma(n)}{\Gamma^2((n + \alpha)/2)} \frac{||F||_{\infty}}{(1 - |z|^2)^\alpha},
\]

where $||F||_{\infty} = \sup\{|F(\zeta) : \zeta \in S^{2n-1}\}$. If $F(\zeta) = \text{const.} \neq 0$ then the inequality is asymptotically sharp as $|z| \to 1 - 0$. 

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Proof. Using Theorem 2 and the series (22) one has

$$\sup_{\|\mathbf{F}\|_\infty=1} |Q_{n,\alpha}(z, F)|(1 - |z|^2)^\alpha = Q_{n,\alpha}(1, 1)$$

$$= \sigma_{n-1} F\left(\frac{n-\alpha}{2}, \frac{n-\alpha}{2}; 1, n, |z|^2\right),$$

where $F(a, b; c; |z|^2)$ is the hypergeometric function.

Since $c - a - b = \alpha > 0$, we have by Gauss’ formula

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Taking $c = n, a = b = (n - \alpha)/2$ and letting $|z| \to 1 - 0$ we obtain

$$Q_{n,\alpha}(1, 1) = \frac{2\pi^n\Gamma(\alpha)}{\Gamma^2((n + \alpha)/2)}$$

(see another proof of (26) in [5, Theorem 4.2.7]).

The equalities (25), (26) and the following consequence of (22)

$$Q_{n,\alpha}(1, 1) \leq \lim_{|z| \to 1-0} Q_{n,\alpha}(1, 1) = Q_{n,\alpha}(1, 1)$$

imply (24) and the asymptotic equality

$$\lim_{|z| \to 1} (1 - |z|^2)^\alpha Q_{n,\alpha}(z, 1) = \frac{2\pi^n\Gamma(\alpha)}{\Gamma^2((n + \alpha)/2)}.$$

These complete the proof of Corollary 2.1. \hfill \Box

References