Some Questions on Metrizability

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Communicated by Rade Živaljević

Abstract. Let us say that a $g$-function $g(n, x)$ on a space $X$ satisfies the condition $(\ast)$ provided: If \( \{x_n\} \to p \in X \) and $x_n \in g(n, y_n)$ for every $n \in \mathbb{N}$, then $y_n \to p$. We prove that a $k$-space $X$ is a metrizable space (a metrizable space with property $ACF$) if and only if there exists a strongly decreasing $g$-function $g(n, x)$ on $X$ such that \( \{g(n, x) : x \in X\} \) is $CF$ for every $n \in \mathbb{N}$ and the condition $(\ast)$ is satisfied. Our results give a partial answer to a question posed by Z. Yun, X. Yang and Y. Ge and a positive answer to a conjecture posed by S. Lin, respectively.

1. Introduction

How to characterize metrizable spaces in terms of $g$-functions is an important question of metrizability. In [6], Z. Yun, X. Yang and Y. Ge gave the following result.

Theorem 1.1. [6, Theorem 4] A Fréchet space $X$ is metrizable if and only if there exists a strongly decreasing $g$-function $g(n, x)$ on $X$ such that \( \{g(n, x) : x \in X\} \) is $CF$ in $X$ for every $n \in \mathbb{N}$ and the following condition is satisfied.

\[(\ast) \quad \text{If } \{x_n\} \to p \in X \text{ and } x_n \in g(n, y_n) \text{ for every } n \in \mathbb{N}, \text{ then } y_n \to p.\]

The authors of [6] noted that the condition “Fréchet” in Theorem 1.1 can be relaxed to “$k'$”, but it cannot be omitted. However, they still do not know whether the condition “Fréchet” in Theorem 1.1 can be relaxed to “$k$”. So they raised the following question.

Question 1.2. [6, Question 1]. For a $k$-space $X$, are the following (1) and (2) equivalent?

1. $X$ is a metrizable space.

2000 Mathematics Subject Classification: 54D50, 54E35.

Key words and phrases: strongly decreasing $g$-function, $CF$-family, metrizable space, $k$-space.

This project was supported by NSF of the Education Committee of Jiangsu Province in China (No.02KJB110001).
Any space is the quotient space of a space with the property \( F \bigcup \{ \) respectively. If also\( \) hereditarily \( CF \),

Notice that \( CF^* \Rightarrow CF \). Taking Question 1.2 into account, Lin [3] raised a conjecture.

**Conjecture 1.3.** [3, Conjecture 1] A \( k \)-space \( X \) is metrizable (with some property) if and only if there exists a strongly decreasing \( g \)-function \( g(n, x) \) on \( X \) such that \( \{ g(n, x) : x \in X \} \) is \( CF^* \) in \( X \) for every \( n \in N \) and the condition \( (\ast) \) is satisfied.

Here we investigate Question 1.2 and Conjecture 1.3. We prove that a \( k \)-space \( X \) is a metrizable space (a metrizable space with property \( ACF \)) if and only if there exists a strongly decreasing \( g \)-function \( g(n, x) \) on \( X \) such that \( \{ g(n, x) : x \in X \} \) is \( CF \) \((g(n, x) : x \in X) \) is \( CF^* \) in \( X \) for every \( n \in N \) and the condition \( (\ast) \) is satisfied. This gives a partial answer to Question 1.2 and a positive answer to Conjecture 1.3. As a corollary of the above results, a space \( X \) is a metrizable space with property \( ACF \) if and only if there exists a strongly decreasing \( g \)-function \( g(n, x) \) on \( X \) such that \( \{ g(n, x) : x \in X \} \) is \( HCP \) in \( X \) for every \( n \in N \) and the condition \( (\ast) \) is satisfied.

Throughout this paper, all spaces are assumed to be regular. \( N \) and \( \omega \) denote the set of all natural numbers and the first infinite ordinal, respectively. For a set \( A \), \( |A| \) denotes the cardinality of \( A \). Let \( A \) be a subset of a space \( X \) and let \( \mathcal{F} \) be a family of subsets of \( X \). \( \overline{A}, \overline{\mathcal{F}}, \bigcup \mathcal{F} \) and \( A \wedge \mathcal{F} \) denote the closure of \( A \), the family \( \{ \overline{F} : F \in \mathcal{F} \} \), the union \( \bigcup \{ F : F \in \mathcal{F} \} \) and the family \( \{ A \cap F : F \in \mathcal{F} \} \), respectively. If also \( x \in X \), \( (\mathcal{F})_x \) denotes the subfamily \( \{ F \in \mathcal{F} : x \in F \} \) of \( \mathcal{F} \) and \( \bigcup (\mathcal{F})_x \) is replaced by \( st(x, \mathcal{F}) \). One may consult [1] for undefined notation and terminology.

**Definition 1.4.** [5] A space \( X \) is said to have property \( ACF \) if every compact subset of \( X \) is finite.

**Remark 1.5.** Any space is the quotient space of a space with the property \( ACF \) [5].

**Definition 1.6.** [6] Let \( X \) be a space and let \( \tau \) be the topology on \( X \). A function \( g : N \times X \to \tau \) is called a \( g \)-function on \( X \) (we write \( g(n, x) \) for short) if \( x \in g(n, x) \) for every \( n \in N \) and every \( x \in X \). A \( g \)-function \( g(n, x) \) on \( X \) is called strongly decreasing if \( g(n + 1, x) \subset g(n, x) \) for every \( n \in N \) and every \( x \in X \).

**Definition 1.7.** [4] Let \( \mathcal{F} \) be a family of subsets of a space \( X \). \( \mathcal{F} \) is called \( CF \) in \( X \) if for every compact subset \( K \subset X \), \( K \wedge \mathcal{F} = \{ F_1, F_2, \ldots, F_k \} \), that is, \( |K \wedge \mathcal{F}| < \omega \), and called \( CF^* \) in \( X \) if also only finitely many \( F \in \mathcal{F} \) have infinite intersections with \( K \). \( \mathcal{F} \) is called closure-preserving in \( X \) if for every subfamily \( \mathcal{F}' \) of \( \mathcal{F} \), \( \bigcup \mathcal{F}' = \bigcup \overline{\mathcal{F}'} \). \( \mathcal{F} = \{ F_\alpha : \alpha \in A \} \) is called hereditarily closure-preserving (hereditarily \( CF \), hereditarily \( CF^* \)) in \( X \) if for any choice \( E_\alpha \subset F_\alpha \), the family \( \{ E_\alpha : \alpha \in A \} \) is closure-preserving \( (CF, CF^*) \) in \( X \).

Throughout this paper, we use brief notations for the following terms.
Let $H \subseteq X$. It is well known that Fréchet $\Rightarrow k' \Rightarrow k$ and none of the implications can be reversed.

2. The main results

We start by giving some lemmas.

**Lemma 2.1.** [6, Lemma 1] Let $\mathcal{F}$ be a family of subsets of a space $X$. If $\mathcal{F}$ is $CF$ in $X$, then $\left\{ \bigcup \mathcal{F}' : \mathcal{F}' \subseteq \mathcal{F} \right\}$ is also $CF$ in $X$.

**Lemma 2.2.** (1) For a family of subsets of a space, locally finite $\Rightarrow HCP \Rightarrow CP$, $HCP \Rightarrow CF^* \Rightarrow CF$ [4, Proposition 3.7] and $HCF \Leftrightarrow CF^* \Leftrightarrow HCF^*$ [2, Theorem 1].

(2) For a family of closed subsets of a $k'$-space, $CF \Rightarrow CP$ [2, Lemma 2] and $CF^* \Leftrightarrow HCP$ [2, Theorem 4].

(3) For a family of subsets of a $k'$-space, $CF^* \Leftrightarrow HCP$ [2, Theorem 6].

**Lemma 2.3.** [6, Theorem 3] A space $X$ is metrizable if and only if there exists a $CP$ strongly decreasing $g$-function $g(n, x)$ on $X$ such that the condition $(*)$ is satisfied.

The proof of the following lemma is trivial, and we omit it.

**Lemma 2.4.** Let $\mathcal{F}$ be a family of subsets of a space with property $ACF$. Then $\mathcal{F}$ is $CF^*$ in $X$.

**Lemma 2.5.** Let $X$ be a space. If there exists a $CF^*$ $g$-function $g(n, x)$ on $X$, then $X$ has property $ACF$.

**Proof.** Let $K$ be a compact subset of $X$ and let $n \in N$. Then $\{g(n, x) : x \in X\}$ is $HCF$ in $X$ from Lemma 2.2(1). Note that $\{x\} \subseteq g(n, x)$ for every $x \in X$.

$\{x : x \in X\}$ is $CF$ in $X$, so $\{x : x \in K\} = K \cap \{x : x \in X\}$ is finite, that is, $K$ is finite. This proves that $X$ has property $ACF$. □

The following theorem gives an almost positive answer to Question 1.2.

**Theorem 2.6.** A $k'$-space $X$ is metrizable if and only if there exists a $\overline{CF}$ strongly decreasing $g$-function $g(n, x)$ on $X$ such that the condition $(*)$ is satisfied.
Proof. Necessity: Assume that $X$ is a metrizable space. We denote the diameter of subset $A$ of $X$ by $d(A)$. Let $\mathcal{U}_1$ be a locally finite cover of $X$ such that $d(U) < 1$ for every $U \in \mathcal{U}_1$. Put $g(1, x) = st(x, \mathcal{U}_1)$ for every $x \in X$. Let $\mathcal{U}_2$ be a locally finite cover of $X$ such that $d(U) < 1/2$ for every $U \in \mathcal{U}_2$, and $\{U : U \in \mathcal{U}_2\}$ is a refinement of $\mathcal{U}_1$. Put $g(2, x) = st(x, \mathcal{U}_2)$ for every $x \in X$. Generally, Let $\mathcal{U}_n$ be a locally finite cover of $X$ such that $d(U) < 1/n$ for every $U \in \mathcal{U}_n$, and $\{U : U \in \mathcal{U}_n\}$ is a refinement of $\mathcal{U}_{n-1}$. Put $g(n, x) = st(x, \mathcal{U}_n)$ for every $x \in X$.

Thus we obtain a $g$-function $g(n, x)$ on $X$. By the proof of (a) $\Rightarrow$ (b) in [6, Theorem 3], the $g$-function $g(n, x)$ is strongly decreasing and satisfies the condition $(\ast)$. Now we only need to prove that $\{g(n, x) : x \in X\}$ is $CF$ in $X$ for every $n \in N$.

In fact, for every $n \in N$, $\mathcal{U}_n$ is locally-finite, so $\overline{\mathcal{U}_n}$ is locally finite. Put $F(n, x) = \bigcup \{U \in \mathcal{U}_n : x \in U\}$. Then $\{F(n, x) : x \in X\}$ is $CF$ in $X$ by Lemma 2.2(1) and Lemma 2.1. For every $x \in X$, note that $(\mathcal{U}_n)_x$ is a finite subfamily of $\mathcal{U}_n$. $F(n, x) = \bigcup \{U \in \mathcal{U}_n : x \in U\} = \bigcup \{U \in \mathcal{U}_n : x \in U\} = st(x, \mathcal{U}_n)$. Thus $g(n, x) = st(x, \mathcal{U}_n) = F(n, x)$. So $\{g(n, x) : x \in X\}$ is $CF$ in $X$.

Sufficiency: Let $X$ be a $k$-space. Assume that there exists a $CF$ strongly decreasing $g$-function $g(n, x)$ on $X$ such that the condition $(\ast)$ is satisfied. Then for every $n \in N$, $\{g(n, x) : x \in X\}$ is $CF$ in $X$. Since $X$ is a $k$-space, $\{g(n, x) : x \in X\}$ is $CP$ in $X$ by Lemma 2.2(2). Note that a family $\mathcal{F}$ of subsets of a space is $CP$ in $X$ if and only if $\overline{\mathcal{F}}$ is $CP$ in $X$. $\{g(n, x) : x \in X\}$ is $CP$ in $X$, so there exists a $CP$ strongly decreasing $g$-function $g(n, x)$ on $X$ such that the condition $(\ast)$ is satisfied. Thus $X$ is a metrizable space by Lemma 2.3.}

The following theorem gives a positive answer to Conjecture 1.3.

**Theorem 2.7.** A $k$-space $X$ is a metrizable space with property $ACF$ if and only if there exists a $CF^*$ strongly decreasing $g$-function $g(n, x)$ on $X$ such that the condition $(\ast)$ is satisfied.

**Proof.** Necessity: Assume that $X$ is a metrizable space with property $ACF$. By Theorem 2.6, there exists a strongly decreasing $g$-function $g(n, x)$ on $X$ such that the condition $(\ast)$ is satisfied. Since $X$ has property $ACF$, $\{g(n, x) : x \in X\}$ is $CF^*$ in $X$ by Lemma 2.4 for every $n \in N$. So there exists a $CF^*$ strongly decreasing $g$-function $g(n, x)$ on $X$ such that the condition $(\ast)$ is satisfied.

Sufficiency: Let $X$ be a $k$-space. Assume that there exists a $CF^*$ strongly decreasing $g$-function $g(n, x)$ on $X$ such that the condition $(\ast)$ is satisfied. At first, it is obvious that $X$ has the property $ACF$ by Lemma 2.5. For every $n \in N$, since $\{g(n, x) : x \in X\}$ is $CF^*$ in $X$ and $g(n+1, x) \subset g(n, x)$ for every $x \in X$, $\{g(n+1, x) : x \in X\}$ is $CF$ in $X$ by Lemma 2.2(1). Thus there exists a $CF$ strongly decreasing $g$-function $g'(n, x)$ on $X$ such that the condition $(\ast)$ is satisfied, where $g'(n, x) = g(n+1, x)$. So $X$ is metrizable by Theorem 2.6.

**Corollary 2.8.** A space $X$ is a metrizable space with property $ACF$ if and only if there exists an $HCP$ strongly decreasing $g$-function $g(n, x)$ on $X$ such that the condition $(\ast)$ is satisfied.
Proof. Necessity: Assume that $X$ is a metrizable space with property $ACF$. By Theorem 2.7, there exists a $CF^*$ strongly decreasing $g$-function $g(n, x)$ on $X$ such that the condition (*) is satisfied. Note that $X$ is a $k'$-space. By Lemma 2.2(3), there exists an $HCP$ strongly decreasing $g$-function $g(n, x)$ on $X$ such that the condition (*) is satisfied.

Sufficiency: Assume that there exists an $HCP$ strongly decreasing $g$-function $g(n, x)$ on $X$ such that the condition (*) is satisfied. Since $HCP \Rightarrow CP$ by Lemma 2.2(1), $X$ is metrizable by Lemma 2.3. Since $HCP \Rightarrow CF^*$ by Lemma 2.2(1), there exists a $CF^*$ strongly decreasing $g$-function $g(n, x)$ on $k'$-space $X$ such that the condition (*) is satisfied. Thus $X$ is a metrizable space with property $ACF$ by Theorem 2.7. □

The authors would like to thank the referee for his valuable amendments.

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