ON THE OPERATOR EQUATIONS

$ABA = A^2$ AND $BAB = B^2$

Christoph Schmoeger

Communicated by Stevan Pilipović

Abstract. We generalize a result of I. Vidav concerning the operator equations $ABA = A^2$ and $BAB = B^2$.

1. Introduction

In [7] I. Vidav proved the following result:

Theorem 1.1. Let $H$ be a complex Hilbert space and let $A$ and $B$ be bounded linear operators on $H$. Then the following assertions are equivalent:

(a) There is a uniquely determined bounded linear operator $P$ on $H$ such that $P^2 = P$ and $A = PP^*$ and $B = P^*P$.

(b) $A$ and $B$ are selfadjoint and satisfy the relations $ABA = A^2$ and $BAB = B^2$.

Vidav gave two proofs of Theorem 1.1; the first proof is geometrical and the second one is algebraic. In [6] Rakočević gave another proof of Theorem 1.1.

The aim of this paper is to prove a result, which implies Theorem 1.1. Section 2 deals with Drazin invertible elements of rings. In Section 3 we consider bounded linear operators on Banach spaces. Operators on Hilbert spaces are considered in Section 4, where we will give a proof of Theorem 1.1. In the final section we investigate several special classes of operators.

2. Drazin inverses in rings

In this section $\mathcal{R}$ denotes an associative ring. An element $A \in \mathcal{R}$ is said to be Drazin invertible if there exists $C \in \mathcal{R}$ such that

1. $A^m = A^{m+1}C$ for some integer $m \geq 0$,
2. $C = AC^2$,
3. $AC = CA$.

2000 Mathematics Subject Classification: Primary 47A05.
Key words and phrases: selfadjoint operator, Drazin inverse.
In this case \( C \) is called a **Drazin inverse** of \( A \) and the smallest integer \( m \geq 0 \) in (1) is called the **index** \( i(A) \) of \( A \).

If \( R \) has a neutral element \( I \) and if we define \( A^0 = I \), then (1), (2) and (3) hold with \( m = 0 \) if and only if \( A \) is invertible.

**Proposition 2.1.** If \( A \in R \) is Drazin invertible, then \( A \) has a unique Drazin inverse.

**Proof.** \[4\].

Our main result in this section is:

**Theorem 2.2.** (a) If \( P, Q \in R \), \( P^2 = P, Q^2 = Q, A = PQ \) and \( B =QP \), then (1), (2) and (3) hold with \( m = 0 \) if and only if \( A \) is invertible.

**Proof.** (a) We have \( ABA = P Q^2 P^2 Q = (PQ)^2 = A^2 \) and \( BAB = Q P^2 Q^2 P = (QP)^2 = B^2 \).

(b) Since \( i(A) = i(B) = 1 \), there are \( C, D \in R \) with

\[
AC = A, \quad CAC = C, \quad AC = CA \\
BDB = B, \quad DBD = D, \quad BD = DB.
\]

Let \( P := CAB, Q := BAC \) and \( R := DBA \). Then

\[
P^2 = CABCA = C(ABA)CB = CA^2 CB = ACACB = ACB = CAB = P \\
R^2 = DBADBA = D(BAB)DA = DB^2 DA = BDBDA = BDA = DBA = R \\
Q^2 = BACBAC = BC(ABA)C = BCA^2 C = BCACA = BCA = BAC = Q.
\]

Furthermore we have

\[
PQ = CABBAC = CAB^2 AC = CA(BAB)AC = C(ABA)BAC \\
= CA^2 BAC = ACABAC = ABAC = A^2 C = ACA = A, \\
RP = DB(ACA)B = DBAB = DB^2 = DBD = B.
\]

It follows that

\[
QP = BACCB = B(ACA)CB = BACB = BCAB \\
= BP = (RP)P = RP^2 = RP = B. \]

### 3. Bounded linear operators

In this section \( X \) denotes a complex Banach space and \( L(X) \) the Banach algebra of all bounded linear operators on \( X \). If \( A \in L(X) \), then \( \sigma(A) \), \( \rho(A) \) and \( r(A) \) denote the spectrum, the resolvent set and the spectral radius of \( A \), respectively. We write \( N(A) \) for the kernel of \( A \) and \( A(X) \) for the range of \( A \). Define \( p(A) \) [resp. \( q(A) \)], the **ascent** [resp. the **descent**] of \( A \), to be the smallest integer \( n \geq 0 \) such that \( N(A^{n+1}) = N(A^n) \) [resp. \( A^{n+1}(X) = A^n(X) \)] or \( \infty \) if no such \( n \) exists. It follows
from [5, Satz 72.3] that if \( p(A) \) and \( q(A) \) are both finite, then they are equal and, if \( p = p(A) = q(A) < \infty \), then \( X = N(Ap) \oplus Ap(X) \).

A Drazin invertible operator \( A \in \mathcal{L}(X) \) with \( i(A) \leq 1 \) is called simply polar.

The following proposition tells us exactly which operators are Drazin invertible.

**Proposition 3.1.** For \( A \in \mathcal{L}(X) \) and \( n \geq 1 \) the following assertions are equivalent:

(a) \( A \) is Drazin invertible and \( i(A) = n \).

(b) \( p(A) = q(A) = 1 \).

(c) \( \text{The resolvent } (\lambda I - A)^{-1} \text{ has a pole of order } n \text{ at } \lambda = 0 \).

**Proof.** [2, Theorem 5.2], [5, Satz 101.2]. \qed

As an immediate consequence of Proposition 3.1 and Theorem 2.2 we get the main result of this section:

**Theorem 3.2.** Suppose that \( A, B \in \mathcal{L}(X) \), \( p(A) = q(A) = 1 \) and \( p(B) = q(B) = 1 \). Then the following assertions are equivalent:

(a) There are \( P, Q \in \mathcal{L}(X) \) such that \( P^2 = P \), \( Q^2 = Q \), \( A = PQ \) and \( B = QP \).

(b) \( ABA = A^2 \) and \( BAB = B^2 \).

We use \( \sigma_p(A) \), \( \sigma_{ap}(A) \), \( \sigma_r(A) \) and \( \sigma_c(A) \) to denote the point, approximate point, residual and continuous spectrum of \( A \in \mathcal{L}(X) \), respectively.

**Corollary 3.3.** Suppose that \( A, B \in \mathcal{L}(X) \), \( p(A) = q(A) = p(B) = q(B) = 1 \), \( ABA = A^2 \) and that \( BAB = B^2 \). Then:

(a) \( \sigma(A) = \sigma(B) \);

(b) \( \sigma_p(A) = \sigma_p(B) \);

(c) \( \sigma_{ap}(A) = \sigma_{ap}(B) \);

(d) \( \sigma_r(A) = \sigma_r(B) \);

(e) \( \sigma_c(A) = \sigma_c(B) \).

**Proof.** Recall that \( \sigma_p(A) \), \( \sigma_r(A) \) and \( \sigma_c(A) \) are pairwise disjoint and that their union is \( \sigma(A) \). Thus (a) follows from (b), (d) and (e).

(b) Since \( p(A) = p(B) > 0 \), \( 0 \in \sigma_p(A) \) and \( 0 \in \sigma_p(B) \). From [1, Theorem 3] and Theorem 3.2 we get

\[
\sigma_p(A) \setminus \{0\} = \sigma_p(PQ) \setminus \{0\} = \sigma_p(QP) \setminus \{0\} = \sigma_p(B) \setminus \{0\}
\]

hence \( \sigma_p(A) = \sigma_p(B) \).

(c) Because of \( \sigma_p(A) \subseteq \sigma_{ap}(A) \) and \( \sigma_p(B) \subseteq \sigma_{ap}(B) \), it follows that \( 0 \in \sigma_{ap}(A) \) and \( 0 \in \sigma_{ap}(B) \). As in the proof of (b) we see with Theorem 3 in [1] that \( \sigma_{ap}(A) = \sigma_{ap}(B) \).

(d) Since \( 0 \in \sigma_p(A) \) and \( 0 \in \sigma_p(B) \), \( 0 \notin \sigma_r(A) \) and \( 0 \notin \sigma_r(B) \). Proceed as in the proof of (b), to obtain \( \sigma_r(A) = \sigma_r(B) \).

(e) Similar. \qed

An operator \( A \in \mathcal{L}(X) \) is called a Fredholm operator if \( \dim N(A) < \infty \) and \( \text{codim } A(X) < \infty \). In this case we set \( \text{ind}(A) = \dim N(A) - \text{codim } A(X) \).
By \( \mathcal{F}(X) \) we denote the ideal of all finite dimensional operators in \( \mathcal{L}(X) \). Let \( \tilde{\mathcal{L}} \) denote the quotient algebra \( \mathcal{L}(X)/\mathcal{F}(X) \) and write \( \tilde{A} \) for the coset \( A + \mathcal{F}(X) \) of \( A \in \mathcal{L}(X) \) in \( \tilde{\mathcal{L}} \). From [5, Satz 81.1] we have

\[ A \text{ is a Fredholm operator } \iff \tilde{A} \text{ is invertible in } \tilde{\mathcal{L}}. \]

**Corollary 3.4.** Let \( A \) and \( B \) as in Corollary 3.3 and \( \lambda \in \mathbb{C} \). Then:

\[ \lambda I - A \text{ is a Fredholm operator } \iff \lambda I - B \text{ is a Fredholm operator}. \]

**Proof.** We first consider the case \( \lambda = 0 \). Let \( A \) be a Fredholm operator, thus \( \tilde{A} \) is invertible in \( \tilde{\mathcal{L}} \). From \( \tilde{A} \tilde{B} \tilde{A} = \tilde{A}^2 \) we obtain \( \tilde{B} = \tilde{I} \), hence \( B \) is a Fredholm operator. Since \( \tilde{B} \tilde{A} \tilde{B} = \tilde{B}^2 \), it follows that \( \tilde{A} = \tilde{I} \). Hence there are \( F_1, F_2 \in \mathcal{F}(X) \) such that \( A = I + F_1 \) and \( B = I + F_2 \). By [5, Satz 81.3],

\[ \text{ind}(A) = \text{ind}(I + F_1) = \text{ind}(I) = 0 = \text{ind}(I + F_2) = \text{ind}(B). \]

Now assume that \( \lambda \neq 0 \). Our statements follow directly from [1, Theorem 6] and Theorem 3.2. \( \square \)

### 4. Operators on Hilbert spaces

In this section we will give a proof of Theorem 1.1. \( H \) denotes a complex Hilbert space. If \( A \in \mathcal{L}(H) \) we write \( \text{iso} \sigma(A) \) for the set of all isolated points of \( \sigma(A) \).

**Proposition 4.1.** Let \( A \in \mathcal{L}(H) \) be normal and \( 0 \in \text{iso} \sigma(A) \).

(a) \( 0 \) is simple pole of the resolvent \( (\lambda I - A)^{-1} \).
(b) \( p(A) = q(A) = 1 \).
(c) \( A \) is Drazin invertible and \( i(A) = 1 \).

**Proof.** (a) follows from [5, Satz 112.2], (b) and (c) follow from Proposition 3.1. \( \square \)

**Theorem 4.2.** Let \( A, B \in \mathcal{L}(H) \) be selfadjoint, \( ABA = A^2 \) and \( BAB = B^2 \).

(a) \( 0 \in \rho(A) \) or \( 0 \) is a simple pole of \( (\lambda I - A)^{-1} \).
(b) \( \sigma(A) \subseteq \{0\} \cup [1, \infty) \) (hence \( A \geq 0 \)).
(c) \( A \) is Drazin invertible and \( i(A) \leq 1 \).
(d) If \( C \) is the Drazin inverse of \( A \), then \( C = C^* \) and \( 0 \leq C \leq I \).
(e) If \( A \neq 0 \), then \( \|A\| \geq 1 \).
(f) If \( \|A\| = 1 \), then \( A^2 = A = B \).

**Proof.** (a) and (b): From

\[
A(B - I)^2 A = A(B^2 - 2B + I)A = AB^2 A - 2ABA + A^2 = A^2 BA - A^2 = A^3 - A^2
\]

it follows that \( A^3 - A^2 = (A(B - I))(A(B - I))^* \geq 0 \), therefore \( \sigma(A^3 - A^2) \subseteq [0, \infty) \).

Now take \( \lambda \in \sigma(A) \setminus \{0\} \). The spectral mapping theorem gives \( \lambda^2(\lambda - 1) = \lambda^3 - \lambda^2 \geq 0 \),
Theorem 2.2 (a) shows that (a) implies (b). Now let 

\[ \lambda \geq 1. \]

This shows (b) and \( 0 \in \text{iso}\, \sigma(A) \) or \( 0 \in \rho(A) \). Now use Proposition 4.1 to derive (a).

(c) follows from (a), (b) and Proposition 4.1.

(d) Because of \( AC = A, CAC = C \) and \( AC = CA \) it follows that \( AC^* A = A \), \( C^* AC^* = C^* A \) and \( AC^* = C^* A \), hence \( C^* \) is a Drazin inverse of \( A \). By Proposition 2.1, \( C = C^* \). If \( 0 \in \rho(A) \), then \( A = I \), thus \( C = I \), hence \( \|C\| = 1 \). Now let \( 0 \in \sigma(A) \). In [2, page 53] it is shown that \( r(C)^{-1} = \text{dist}(0, \sigma(A) \smallsetminus \{0\}) \).

Now we see from (b) that \( r(C)^{-1} \geq 1 \), hence, since \( C = C^* \), \( \|C\| = r(C) \leq 1 \). We denote the inner product on \( H \) by \( \langle \cdot, \cdot \rangle \). Take \( x \in H \) and let \( y = Cx \). Then

\[ \langle Cx, x \rangle = \langle CACx, x \rangle = \langle ACx, Cx \rangle = \langle Ay, y \rangle \geq 0, \]

since \( A \geq 0 \). Thus \( C \geq 0 \). From \( \|C\| \leq 1 \) we obtain \( 0 \leq C \leq I \).

(e) If \( A \neq 0 \), we have \( \|A\| = r(A) \geq 1 \), by (b).

(f) If \( \|A\| = 1 \), then \( r(A) = 1 \), thus we obtain from (b) that \( \sigma(A) \subseteq \{0, 1\} \). By the spectral mapping theorem, \( \sigma(A^2 - A) = \{0\} \), hence \( \|A^2 - A\| = r(A^2 - A) = 0 \), this gives \( A^2 = A \). Since \( \sigma(A) = \sigma(B) \) (Corollary 3.3), we see that \( \|B\| = r(B) = r(A) = \|A\| = 1 \). Hence, by the same arguments as above, \( B^2 = B \). It follows that \( ABA = A \) and \( BAB = B \), hence \( \|A\|^2 = A \|B\|^2 = B \). A projection \( A \) is Drazin if \( \|A\| = 1 \). From [8, Satz V.5.9] we derive that \( ABA = BA \). Hence \( AB = BA \). We conclude that \( A = ABA = BA^2 = BA = B^2A = BAB = B \).

Proof of Theorem 1.1. Theorem 2.2 (a) shows that (a) implies (b). Now suppose that (b) is valid. If \( 0 \in \rho(A) \), then \( A = B = I \) and we are done. Therefore we can assume that \( 0 \in \sigma(A) \) and \( 0 \in \sigma(B) \). By Theorem 4.2, \( A \) and \( B \) are Drazin invertible and \( \sigma(A) = \sigma(B) \). Let \( P \) and \( Q \) as in the proof of Theorem 2.2 (b).

Hence \( P = CAB, Q = BAC, PQ = A, QP = B \) and \( C \) is the Drazin inverse of \( A \). From Theorem 4.2 we get \( C = C^* \), thus \( P^* = BAC = Q \).

It remains to show that \( P \) is uniquely determined. Suppose that \( R^2 = R, PP^* = RR^* \) and \( P^*P = R^*R \). Then \( P^*P(I-R) = R^*R(I-R) = R^*R - R^*R = 0 \), thus \( P(I-R)x \subseteq N(P)+ = P(x)^\perp \), hence \( P(I-R) = 0 \). Therefore we have \( PR = P \). A similar argument gives \( R^*P^* = R^* \). Taking adjoints we obtain \( R = PR = P \).

Proof of Theorem 1.1.

5. Examples and remarks

In this section we give some examples of operators \( A \) which are Drazin invertible with \( i(A) = 1 \). \( X \) always denotes a complex Banach space.

An operator \( A \in \mathcal{L}(X) \) is called hermitian if \( \|\exp(itA)\| = 1 \) for all \( t \in \mathbb{R} \).

Example 5.1. If \( A \in \mathcal{L}(X) \) is hermitian and if \( 0 \in \text{iso}\, \sigma(A) \), then \( A \) is Drazin invertible and \( i(A) = 1 \).

Proof. Let \( P_0 \) be the spectral projection associated with \{0\}. Let \( M_0 = P_0(X) \) and \( A_0 = A|_{M_0} \). By [5, Satz 100.1] we have \( A(M_0) \subseteq M_0 \) and \( \sigma(A_0) = \{0\} \). Since \( A_0 \) is hermitian operator on \( M_0 \) [3, Proposition 4.12], we have \( \|A_0\| = r(A_0) = 0 \) [3, Theorem 4.10]. It follows that \( AP_0 = 0 \). Now [5, (101.9)] shows that 0 is a simple pole of \( (\lambda I - A)^{-1} \). Proposition 3.1 completes the proof.
An operator $A \in \mathcal{L}(X)$ is said to be \textit{paranormal} if $\|Ax\|^2 \leq \|A^2x\| \|x\|$ for all $x \in X$.

**Example 5.2.** If $A \in \mathcal{L}(X)$ is paranormal and if $0 \in \sigma(A)$, then $A$ is Drazin invertible and $i(A) = 1$.

**Proof.** Let $P_0, M_0$ and $A_0$ as in the proof of 5.1. From [5, page 500] we get $\|A_0\| = r(A_0) = 0$. Now proceed as in the proof of 5.1. □

A bounded linear operator $A$ on a Hilbert space $H$ is called \textit{hyponormal} if $\|A^*x\| \leq \|Ax\|$ for all $x \in H$. Since hyponormal operators are paranormal, we have by Example 5.2:

**Example 5.3.** If $A \in \mathcal{L}(H)$ is hyponormal and if $0 \in \sigma(A)$, then $A$ is Drazin invertible and $i(A) = 1$.

**Remark 5.4.** If $A, B \in \mathcal{L}(X)$, $ABA = A^2$, $BAB = B^2$, $AB = BA$, $p(A) \leq 1$ and $p(B) \leq 1$, then $A^2 = A = B$.

**Proof.** From $A^2 = A^2B = ABA = AB^2 = B^2$ it follows that $A^3 = AB^2 = A^2$, thus $A^2(A - I) = 0$. Since $p(A) \leq 1$, we get $A(A - I) = 0$, hence $A^2 = A$. In the same way we derive $B^2 = B$. Consequently


**Remark 5.5.** Suppose that $A, B \in \mathcal{L}(X)$ are paranormal, $ABA = A^2$, $BAB = B^2$ and $AB = BA$; then $A^2 = A = B$.

**Proof.** Since $\|Ax\|^2 \leq \|A^2x\| \|x\|$ for $x \in X$, it follows that $p(A) \leq 1$. Similarly $p(B) \leq 1$. Now use 5.4. □

**Remark 5.6.** Suppose that $H$ is a complex Hilbert space, $A, B \in \mathcal{L}(H)$ are normal, $ABA = A^2$, $BAB = B^2$ and $AB = BA$. Then $A$ is selfadjoint and $A^2 = A = B$.

**Proof.** Since normal operators are paranormal, it follows from 5.5 that $A$ and $B$ are normal projections, hence they are selfadjoint. □

**Remark 5.7.** If $A \in \mathcal{L}(X)$ is hermitian, then $p(A) \leq 1$.

**Proof.** Let $x \in N(A^2)$ and $\|x\| = 1$. Then for $t \in \mathbb{R}$,

$1 = \|x\| = \|\exp(-itA)\exp(itA)x\| \leq \|\exp(-itA)\| \|\exp(itA)x\|$

$= \|\exp(itA)x\| \leq \|\exp(itA)\| \|x\| = \|x\| = 1,$

thus, since $A^n x = 0$ for $n \geq 2$,

$1 = \|\exp(itA)x\| = \|x + itAx\|.$

Therefore $|t| \|Ax\| - 1 \leq 1$ for all $t \in \mathbb{R}$. This gives $x \in N(A)$. □

**Remark 5.8.** Suppose that $A, B \in \mathcal{L}(X)$ are hermitian, $ABA = A^2$, $BAB = B^2$ and that $AB = BA$; then $A^2 = A = B$.

**Proof.** 5.7 and 5.4. □
ON THE OPERATOR EQUATIONS $ABA = A^2$ AND $BAB = B^2$

References


Mathematisches Institut I  (Received 16 08 2005)
Universität Karlsruhe (TH)
Englerstraße 2
76128 Karlsruhe
Germany
christoph.schmoeger@math.uni-karlsruhe.de