A 2-DIMENSIONAL ALGEBRAIC VARIETY
WITH 27 RECTILINEAR GENERATORS
AND 108 TRISECANTS
AND ITS CONNECTION WITH THE MAXIMAL
EXCEPTIONAL SIMPLE LIE GROUP

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Abstract. A 2-dimensional algebraic variety in 4-dimensional projective space
determining a regular configuration is considered and its connection with sim-
ples exceptional Lie group $E_8$ is found.

1. Galois and Weyl groups
connected with simple Lie groups

With every complex and real compact simple Lie group two finite groups called
“Galois and Weyl groups” are connected. Galois group $\Gamma$ of a simple Lie group is
the Galois group of characteristic equation of this group, Weyl group $W$ of this Lie
group is the group generated by reflections determined by root vectors of this Lie
group (see [1, pp. 63 and 69]).

É. Cartan [2] has proved that for simple Lie groups $B_n$, $C_n$, $G_2$, $F_4$, $E_7$ and $E_8$
groups $V$ and $W$ are isomorphic, and for simple Lie groups $A_n$, $D_n$, and $E_6$ group
$W$ is an invariant subgroup of group $\Gamma$, and the quotient group $\Gamma/W$ for group $D_4$
consists of 3! elements, for other groups $A_n$, $D_n$, and $E_6$ consists of 2 elements.

Note that the Dynkin diagrams for simple Lie groups with isomorphic $\Gamma$ and
$W$ have no symmetry, for group $D_4$ has 3-lateral symmetry, for other groups $A_n$, $D_n$,
and $E_6$ have 2-lateral symmetries (see [1, pp. 70 and 72]).

Weyl groups of simple Lie groups $A_n$, $B_n$, $C_n$, $D_n$, $G_2$, and $F_4$ are isomor-
phic to the groups of symmetries of $n$-dimensional regular simplex, and the $n$-
dimensional cube and cocube, and $n$-dimensional semicube, regular plane hexagon,
and 4-dimensional bicube, respectively (see [1, p. 71]).

É. Cartan [3, pp. 35–43 and 43–50] has established that the Galois group of
simple Lie group $E_6$ is isomorphic to the group of symmetries of a smooth cubic

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surface $S_3$ in projective space $P^3$ with 27 rectilinear generators and that the Galois group of simple Lie group $E_7$ is isomorphic to the group of symmetries of quartic curve $C_4$ in plane $P^2$ with 28 double tangents (see [1, p. 72]).

The equation of surface $S_3$ can be reduced to the form $ace = bdf$, where $a, b, c, d, e, f$ are linear polynomials. If we denote the intersection of two planes $a = 0$ and $b = 0$ by $ab$, then 9 of these 27 rectilinear generators are lines: $ab, ad, af, cb, cd, ef, eb, ed, ef$. If we denote the 3 rectilinear generators which meet the lines $ab, cd, ef$ by $(ab, cd, ef)_i, i = 1, 2, 3$, etc., then the other 18 rectilinear generators are $(ab, cd, ef)_i, (ad, cf, eb)_i, (af, cb, ed)_i, (ab, cf, cd)_i, (ad, cb, ef)_i$, and $(af, cd, eb)_i$.

The Galois group of the simple Lie group $E_7$ also admits the 3-dimensional geometric interpretation as the group of symmetries of a quartic surface $S_4$ in space $P^3$ with 16 rectilinear generators. The equation of this surface can be reduced to the form $aceg = bdgh$. The 16 rectilinear generators of this surface are straight lines $ab, ad, af, ah, be, be, bg, cd, cf, ch, de, dg, ef, eh, fg, gh$ which are lines of intersection of two tetrahedra bounded by planes $a = 0, c = 0, e = 0, g = 0$ and $b = 0, d = 0, f = 0, h = 0$. The configuration of 28 double tangents of curve $C_4$ can be obtained from the configuration of 16 rectilinear generators of surface $S_4$ and of 12 lines $ac, ae, ag, bd, bf, bh, ca, cg, df, dh, eg, fh$ being edges of mentioned tetrahedra by the projection from a point onto a plane.

É. Cartan [4, pp. 45 and 43] has found that the numbers of elements of these groups are $9!/7$ and $12!/11.5.3.2$, respectively. These groups are subgroups of groups of permutations of 9 rectilinear generators of surface $S_3$ and of 12 edges of mentioned tetrahedra.

2. Algebraic variety $V^2$ and Weyl group of simple Lie group $E_8$

Let us consider 2-dimensional algebraic variety $V^2$ in projective space $P4$ being the intersection of two smooth cubic hypersurfaces whose equations can be reduced to the form $adg = beh = cfi$, where $a, b, c, d, e, f, g, h, i$ are linear polynomials. If we denote the intersection of three hyperplanes $a = 0, b = 0$, and $c = 0$ by $abc$, then 27 rectilinear generators of this variety are lines: $abc, ahf, abi, acc, aef, aei, ahe, ahf, ahi, dbc, dbf, dbi, dec, def, deh, dhc, dfh, dhi, gbe, gbf, gbi, gce, gcf, gei, ghc, ghf, ghj$.

Since the dimension $d$ of the intersection of $i$-dimensional and $j$-dimensional planes in a projective space and the dimension $s$ of the plane generated by these two planes are connected by the equality

$$i + j = d + s$$

(see [1, p. 8]), for every given 3 skew straight lines in 4-dimensional space $P^4$ there are 3 trisecants, that is 3 straight lines meeting all given 3 lines, because two from given 3 lines generate a 3-dimensional plane which according to equality (1) meets the third given line at one point, and through this point one of 3 trisecants passes. If we denote the 3 rectilinear trisecants which meet the lines $abc, def$, and $ghi$ by
Thus on variety $V^2$ there are $27$ rectilinear generators and $108$ rectilinear trisecants.

The surface $S_3$ can be obtained from the variety $V^2$ by the projection from a point onto a hyperplane, this projection maps rectilinear generators and trisecants on variety $V^2$ to rectilinear generators on surface $S_3$.

The method used by É. Cartan in [3] allows to establish that Weyl group of simple Lie group $E_8$ is isomorphic to the group of symmetries of variety $V^2$. This group is isomorphic also to the group of symmetries of polyhedron defined by H. S. M. Coxeter [5]. This polyhedron has 240 vertices, 16 of these vertices are represented by 8 octonion basis elements and their products by $-1$, $224 = 14 \cdot 16$ other vertices are represented by linear combinations of these elements (see [1, pp. 71–71]).

Coxeter [5] has established that the number of elements of this group is equal to $16!/13 \cdot 11 \cdot 7 \cdot 5 \cdot 3 \cdot 2$. This group is a subgroup of group of permutations of 16 vertices of mentioned polyhedron.

3. Geometric interpretations of compact simple Lie groups $E_6$, $E_7$, and $E_8$

In book [1, pp. 340–350] following geometric interpretations of compact simple Lie groups $E_6$, $E_7$, and $E_8$ are presented: they are groups of motions in Hermitian elliptic planes over tensor products of alternative division algebra $O$ of octonions by division algebras $C$ of complex numbers, $H$ of quaternions, and $O$, respectively.

The coordinates $x^0$, $x^1$, $x^2$ of points in these planes are triples of elements in mentioned tensor products belonging to one associative subalgebra of these tensor products determined by two arbitrary elements of these algebras and defined up to right multiplier which is not a zero divisor from the same subalgebra. Straight lines in these planes are determined by equations

$$u_0 x^0 + u_1 x^1 + u_2 x^2 = 0,$$

where coefficients $u_i$ are triples of elements in mentioned tensor products belonging to one associative subalgebra of these tensor products determined by two arbitrary elements of these algebras and defined up to left multiplier which is not a zero divisor from the same subalgebra.
The transformations of coordinates $x^1$ under motions in mentioned planes are not linear representations, the infinitesimal transformations in Lie algebras of these groups were found by Vinberg [6] in 1964 (see [1, pp. 79–80]). In book [1, pp. 342 and 346] was proved that straight lines in mentioned planes admit interpretations as the manifolds of straight lines in real elliptic space $S^9$, of 3-dimensional planes in $S^{11}$, and of 7-dimensional planes in $S^{15}$, respectively.

4. Connection between variety $V^2$ and Hermitian elliptic plane over the tensor product of two algebras $O$

Vinberg [7] has proved in 2003 that two straight lines in the plane over the tensor product of two algebras $O$ meet at 135 points. This result can be also obtained as follows: if equations of two straight lines are $x^0 = 0$ and $x^1 = 0$, the coordinates of common points of these lines are 0, 0, $x^2$. If $x^2$ is not a zero divisor, it can be divided by itself and reduced to 1. This is impossible, if $x^2$ is a zero divisor, and in this case there are many common points of the lines with different coordinates $x^2$.

From paper [7] following results follow: in a straight line over tensor product of two algebras $O$ the finite group of transformations consisting of $256 = 2^8$ elements and being a subgroup of Weyl group of simple Lie group $E_8$ corresponding to transformations of 16 vertices of Coxeter’s polyhedron represented by octonion basis elements and their products by $-1$ can be determined. This group contains identity $E$, 135 involutive elements which are reflections in common points of considered line with another line, and 120 other involutive elements. If points of considered straight line are represented by 7-dimensional planes in space $S^{15}$, elements of mentioned group are represented by reflections in 7-dimensional planes and their polars, 120 other involutive elements of this group are represented by symmetries in paratactic congruences of straight lines in space $S^{15}$ [1, p 269].

Equations (2) imply that coordinates $x^i$ of all common points of two lines in mentioned planes belong to one associative subalgebra of the tensor product of two algebras $O$.

The coincidence of the number 135 of common points of two straight lines in Hermitian elliptic plane over the tensor product of two algebras $O$ with the number $27 + 108$ of rectilinear generators and trisecants on variety $V^2$ shows that it is possible to establish the connection between variety $V^2$ and mentioned Hermitian elliptic plane, and that the groups of symmetries of the configuration of 135 common points of two straight lines in mentioned plane and of variety $V^2$ are isomorphic. Since maximal associative subalgebra in algebra $O$ is isomorphic to algebra $H$ of quaternions, the maximal associative subalgebra of the tensor product of two algebras $O$ is isomorphic to the tensor product of two algebras $H$ which is isomorphic to the algebra of real $(4 \times 4)$-matrices and is 16-dimensional. Therefore straight lines over this tensor product are also 16-dimensional.

The 7-dimensional planes in space $S^{15}$ can be determined by affine matrix coordinates (see [1, p. 133]) which are real $(8 \times 8)$-matrices. These matrices determining 7-dimensional planes representing 135 common points with coordinates 0, 0, $x^2$ consist of four $(4 \times 4)$-submatrices: the upper left submatrix is the unit
(4 × 4)-matrix, the upper right and lower left submatrices are zero (4 × 4)-matrices, the lower right submatrix is real (4 × 4)-matrix representing element $x^2$ of tensor product of two algebras $H$.

Dimension $N$ of manifold of $m$-dimensional planes in $n$-dimensional projective space $P^n$ is equal to

$$N = (m + 1)(n - m)$$

(see [1, p. 134]). According to equality (3), 135 7-dimensional planes in space $S^{15}$ representing 135 common points of a straight line in Hermitian elliptic plane over the tensor product of two algebras $O$ with another straight line in this plane lie in 9-dimensional plane in space $S^{15}$, because $16 = (7 + 1)(9 - 7)$. If this 9-dimensional plane is regarded as space $S^9$, for every 7-dimensional plane in this space there is a polar straight line.

According to equality (1), the intersection of every two 7-dimensional planes in space $S^9$ is 5-dimensional. Since common points of two straight lines in a Hermitian elliptic plane over the tensor product of two algebras $O$ lie on two straight lines, every two of these points are adjacent (see [1, p. 118]). Therefore all 7-dimensional planes in space $S^9$ representing 135 common points of two straight lines in mentioned plane pass through the same 4-dimensional plane corresponding to right (4 × 4)-submatrices of the affine matrix coordinates of 7-dimensional planes representing common points of two straight lines.

The polar straight lines of 135 7-dimensional planes lie in the 4-dimensional plane polar to mentioned 4-dimensional plane. Thus we obtain in this 4-dimensional plane the configuration of 135 straight lines.

Since the group of symmetries of this configuration is isomorphic to the group of symmetries of variety $V^2$, the lines of this configuration are rectilinear generators and trisecants of variety $V^2$.

References


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