UNIFORM DISTRIBUTION MODULO 1 AND
THE UNIVERSALITY OF ZETA-FUNCTIONS
OF CERTAIN CUSP FORMS

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Abstract. An universality theorem on the approximation of analytic func-
tions by shifts $\zeta(s+i\tau, F)$ of zeta-functions of normalized Hecke-eigen forms $F$, where $\tau$ takes values from the set $\{k^\alpha h : k = 0, 1, 2, \ldots \}$ with fixed $0 < \alpha < 1$ and $h > 0$, is obtained.

1. Introduction

Denote by $SL(2, \mathbb{Z})$ the full modular group, i.e., $SL(2, \mathbb{Z}) = \{ (a \ b \\
\ c \ d) : a, b, c, d \in \mathbb{Z}, \ ad - \ bc = 1 \}$. The function $F(z)$ is called a holomorphic cusp form of weight $\kappa$ for $SL(2, \mathbb{Z})$ if $F(z)$ is holomorphic in the half-plane $\text{Im}z > 0$, for all $\begin{pmatrix} a & b \\
\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ satisfies the functional equation

$$F\left(\frac{az + b}{cz + d}\right) = (cz + d)^\kappa F(z),$$

and at infinity has the Fourier series expansion $F(z) = \sum_{m=1}^{\infty} c(m) e^{2\pi imz}$. Assume additionally that $F(z)$ is a normalized Hecke-eigen form, i.e., is an eigen form of all Hecke operators

$$T_m F(z) = m^{\kappa - 1} \sum_{a,d > 0 \atop ad = m} \frac{1}{d^\kappa} \sum_{b \text{ (mod } d)} F\left(\frac{az + b}{d}\right), \quad m \in \mathbb{N},$$

and $c(1) = 1$.

The associated zeta-function $\zeta(s, F)$, $s = \sigma + it$, is defined, for $\sigma > \frac{\kappa + 1}{2}$, by the Dirichlet series

$$\zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}.$$
and can be analytically continued to an entire function. Moreover, the function $\zeta(s, F)$ can be written, for $\sigma > \frac{s+1}{2}$, as a product over primes

$$\zeta(s, F) = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where $\alpha(p)$ and $\beta(p)$ are conjugate complex numbers satisfying $\alpha(p) + \beta(p) = c(p)$.

The zeta-function $\zeta(s, F)$, as the Riemann zeta-function, Dirichlet $L$-functions, and some other zeta and $L$-functions, is universal in that sense that a wide class of analytic functions can be approximated by shifts $\zeta(s + i\tau, F)$ with some real $\tau$. This was obtained in [6] by using the probabilistic approach and positive density method. Let $D = D_F = \{s \in \mathbb{C} : \frac{\sigma}{2} < \sigma < \frac{s+1}{2}\}$. Denote by $K = K_F$ the class of compact subsets of the strip $D$ with connected complements, and by $H_0(K)$, $K \in K$, the class of continuous non-vanishing functions on $K$ which are analytic in the interior of $K$. Let $\text{meas}A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then, in [7], the following statement was proved.

**Theorem 1.1.** Suppose that $K \in K$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, F) - f(s)| < \varepsilon \right\} > 0.$$  

Investigations of universality of zeta-functions of cusp forms were continued in [8] and [6], where the analogues of Theorem 1.1 were obtained for zeta-functions attached to new forms and for zeta-functions of primitive normalized Hecke-eigen forms for the Hecke subgroup with character, respectively.

Theorem 1.1 and its generalizations in [8], [6] are of continuous type because the shifts $\tau$ in $\zeta(s + i\tau, F)$ can take arbitrary real values. Also, the discrete universality of zeta-functions is considered. In this case, $\tau$ takes values from some discrete sets. The discrete analogue of Theorem 1.1 was begun to study in [9], and a general result was obtained in [11]. Denote by $\#A$ the cardinality of the set $A$.

**Theorem 1.2.** Suppose that $K \in K$, $f(s) \in H_0(K)$ and $h > 0$ is an arbitrary fixed number. Then, for every $\varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s \in K} |\zeta(s + ikh, F) - f(s)| < \varepsilon \right\} > 0.$$  

In Theorem 1.2 the shift $\tau$ in $\zeta(s + i\tau, F)$ takes values from the arithmetical progression $\{0, h, 2h, \ldots\}$ with difference $h$. It is an interesting problem to prove Theorem 1.2 when $\tau$ takes values from a more complicated discrete set, and the present paper is devoted to the case of the set $\{k^{\alpha}h : k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$, where $h > 0$ and $0 < \alpha < 1$ are arbitrary fixed numbers.

**Theorem 1.3.** Suppose that $K \in K$, $f(s) \in H_0(K)$, and $h > 0$ and $0 < \alpha < 1$ are arbitrary fixed numbers. Then, for every $\varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s \in K} |\zeta(s + ik^{\alpha}h, F) - f(s)| < \varepsilon \right\} > 0.$$
Let $H(G)$ be the space of analytic functions on the region $G \subset \mathbb{C}$ endowed with the topology of uniform convergence on compacta. In [10], Theorem 1.1 was generalized to composite functions $\Phi(\zeta(s, F))$ for some classes of operators $\Phi : H(D) \to H(D)$. Similarly, discrete analogues of Theorem 1.2 for $\Phi(\zeta(s, F))$ were obtained in [11]. Theorem 1.3 also can be rewritten for composite functions. We give only one example. For $a_1, \ldots, a_r \in \mathbb{C}$ and $\Phi : H(D) \to H(D)$, define
\[
H_{\Phi; a_1, \ldots, a_r}(D) = \{ g \in H(D) : g(s) \neq a_j, \ j = 1, \ldots, r \} \cup \{ \Phi(0) \},
\]
\[
S = \{ g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0 \}.
\]

**Theorem 1.4.** Suppose that $\Phi : H(D) \to H(D)$ is a continuous operator such that $\Phi(S) \supset H_{\Phi; a_1, \ldots, a_r}(D)$, and $h > 0$ and $0 < \alpha < 1$ are arbitrary fixed numbers. If $r = 1$, let $K \subset \mathbb{C}$, $f(s) \in H(K)$ and $f(s) \neq a_1$ on $K$. Then, for every $\varepsilon > 0$,
\[
\liminf_{N \to \infty} \frac{1}{N+1} \# \{ 0 \leq k \leq N : \sup_{s \in K} |\Phi(\zeta(s+ik\alpha h, F)) - f(s)| < \varepsilon \} > 0.
\]

Let $K \subset D$ be an arbitrary compact subset, and $f(s) \in H_{\Phi; a_1, \ldots, a_r}(D)$. Then inequality (1.1) holds for any $\varepsilon > 0$.

For example, Theorem 1.4 implies the discrete universality for the functions $e^{\zeta(s, F)}$, $\sin(\zeta(s, F))$, $\cos(\zeta(s, F))$, etc.

2. **Probabilistic limit theorems**

For the proof of Theorem 1.3 we need the weak convergence for
\[
P_N(A) = \frac{1}{N+1} \# \{ 0 \leq k \leq N : \zeta(s+ik\alpha h, F) \in A \}, \quad A \in \mathcal{B}(H(D)),
\]
with explicitly given limit measure. Here the sequel, $\mathcal{B}(X)$ denotes the Borel $\sigma$-field of the space $X$.

Let $\gamma = \{ s \in \mathbb{C} : |s| = 1 \}$ be the unit circle on the complex plane, and $\mathbb{P}$ be the set of all prime numbers. Define $\Omega = \prod_{p \in \mathbb{P}} \gamma_p$ where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. It is well known that the torus $\Omega$, with the product topology and pointwise multiplication, is a compact topological Abelian group. Thus, on a measurable space $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure $m_H$ can be defined, and we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of an element $\omega \in \Omega$ to the circle $\gamma_p$, $p \in \mathbb{P}$. Then we have that $\{ \omega(p) : p \in \mathbb{P} \}$ is a sequence of independent random variables defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. On the latter space, define the $H(D)$-valued random element $\zeta(s, \omega, F)$ by
\[
\zeta(s, \omega, F) = \prod_{p \in \mathbb{P}} \left( 1 - \alpha(p)\omega(p) \right)^{-1} \left( 1 - \beta(p)\omega(p) \right)^{-1} / p^s,
\]
and denote by $P_\zeta$ the distribution of $\zeta(s, \omega, F)$, i.e., for $A \in \mathcal{B}(H(D))$,
\[
P_\zeta(A) = m_H(\omega \in \Omega : \zeta(s, \omega, F) \in A).
\]

**Theorem 2.1.** The measure $P_N$ converges weakly to $P_\zeta$ as $N \to \infty$. Moreover, the support of $P_\zeta$ is the set $S$. 
The proof of Theorem 2.1 is based on individual properties of the sequence \( \{ k^\alpha : k \in \mathbb{N}_0 \} \). We recall that a sequence \( \{ x_k \} \subset \mathbb{R} \) is uniformly distributed modulo 1 if, for each interval \( I = [a, b) \subset [0, 1) \) of length \( |I| \),
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_I(\{ x_k \}) = |I|,
\]
where \( \{ u \} \) denotes the fractional part of \( u \in \mathbb{R} \), and \( \chi_I \) is the indicator function of \( I \).

**Lemma 2.1.** For an arbitrary fixed \( a \neq 0 \) and \( 0 < \alpha < 1 \), the sequence \( \{ k^\alpha a \} \) is uniformly distributed modulo 1.

The lemma is Exercise 3.10 of [4].

**Lemma 2.2.** Suppose that a sequence \( \{ x_k \} \subset \mathbb{R} \) is such that, for every \( a \neq 0 \), the sequence \( \{ x_k a \} \) is uniformly distributed modulo 1. Then the measure \( Q_N \), defined, for \( h > 0 \), by
\[
Q_N(A) = \frac{1}{N+1} \sum_{k=0}^{N} \prod_p p^{-ix_k h_k} \chi_A(\{ x_k \}), \quad A \in \mathcal{B}(\Omega),
\]
converges weakly to the Haar measure \( m_H \) as \( N \to \infty \).

**Proof.** Let \( g_N(k) = \prod_p p^{-ix_k h_k} \) denote the Fourier transform of \( Q_N \), i.e.,
\[
g_N(k) = \int_\Omega \prod_p \omega^h(p) dQ_N,
\]
where only a finite number of integers \( k_p \) are distinct from zero. By the definition of \( Q_N \), we find that
\[
(2.1) \quad g_N(\overline{k}) = \frac{1}{N+1} \sum_{k=0}^{N} \prod_p p^{-ix_k h_k} = \frac{1}{N+1} \sum_{k=0}^{N} \exp \left\{ -ix_k h_k \sum_p k_p \log p \right\}.
\]
It is well known that the set \( \{ \log p : p \in \mathbb{P} \} \) is linearly independent over the field of rational numbers \( \mathbb{Q} \). Therefore, the equality \( \sum_p k_p \log p = 0 \) holds if and only if \( \overline{k} = \overline{0} \). Clearly,
\[
(2.2) \quad g_N(\overline{0}) = 1.
\]
In the case \( \overline{k} \neq \overline{0} \), we have that \( h \sum_p k_p \log p \neq 0 \). Therefore, by the hypothesis on the sequence \( \{ x_k \} \), the sequence
\[
\left\{ \frac{x_k h}{2\pi} \sum_p k_p \log p \right\}
\]
is uniformly distributed modulo 1. Hence, an application of the Weyl criterion together with (2.1) shows that \( \lim_{N \to \infty} g_N(\overline{k}) = 0 \) for \( \overline{k} \neq \overline{0} \). This and (2.2) yield that
\[
(2.3) \quad \lim_{N \to \infty} g_N(\overline{k}) = \begin{cases} 
1 & \text{if } \overline{k} = \overline{0}, \\
0 & \text{if } \overline{k} \neq \overline{0}.
\end{cases}
\]
Since
\[ g(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0, \end{cases} \]
is the Fourier transform of the Haar measure \( m_H \), a continuity theorem for probability measures on compact groups, see, for example, [3], and (2.3) prove the lemma.

Lemma 2.2 for the sequence \( \{k^\alpha\} \) with \( \alpha > 0 \) and \( \alpha \notin \mathbb{N} \) was proved in [2].

For each \( \omega \in \Omega \), extend the function \( \omega(p) \) from the set \( \mathbb{P} \) to the set \( \mathbb{N} \) by
\[ \omega(m) = \prod_{p \mid m} \omega(p), \quad m \in \mathbb{N}. \]
Further, we consider two functions
\[ \zeta_n(s, F) = \sum_{m=1}^{\infty} \frac{c(m)v_n(m)}{m^s} \quad \text{and} \quad \zeta_n(s, \omega, F) = \sum_{m=1}^{\infty} \frac{c(m)\omega(m)v_n(m)}{m^s}, \]
where, for a fixed number \( \sigma_0 > \frac{1}{2} \) and \( m, n \in \mathbb{N} \),
\[ v_n(m) = \exp \{- (m/n)^{\sigma_0}\}. \]
Then the series for \( \zeta_n(s, F) \) and \( \zeta_n(s, \omega, F) \) are absolutely convergent for \( \sigma > \frac{\sigma_0}{2} \).

For \( A \in \mathcal{B}(H(D)) \), define
\[ P_{N,n}(A) = \frac{1}{N+1} \# \{0 \leq k \leq N : \zeta_n(s + ik h, F) \in A\}. \]
Moreover, let the function \( u_n : \Omega \to H(D) \) be given by \( u_n(\omega) = \zeta_n(s, \omega, F) \), and let the probability measure \( \hat{P}_n \) be defined by \( \hat{P}_n = m_H u_n^{-1} \), i.e., for \( A \in \mathcal{B}(H(D)) \),
\[ \hat{P}_n(A) = m_H u_n^{-1}(A) = m_H(u_n^{-1}A). \]

**Lemma 2.3.** Under hypotheses of Lemma 2.2, \( P_{N,n} \) converges weakly to \( \hat{P}_n \) as \( N \to \infty \).

**Proof.** Since the series for \( \zeta_n(s, \omega, F) \) is absolutely convergent for \( \sigma > \frac{\sigma_0}{2} \), we have that the function \( u_n \) is a continuous one. Moreover,
\[ u_n(p^{-ixk h} : p \in \mathbb{P}) = \zeta_n(s + ik h, F). \]
Therefore, \( P_{N,n} = Q_N u_n^{-1} \). This, Lemma 2.2 and Theorem 5.1 of [1] prove the lemma.

For the proof of Theorem 1.1 a limit theorem for
\[ \hat{P}_T(A) \overset{\text{def}}{=} \frac{1}{T} \text{meas} \{ \tau \in [0, T] : \zeta(s + i\tau, F) \in A \}, \quad A \in \mathcal{B}(H(D)), \]
as \( T \to \infty \) was applied. For our propose, we need some facts from the proof of the above limit theorem.
Lemma 2.4. The measure \( \hat{Q}_T \) defined by
\[
\hat{Q}_T(A) = \frac{1}{T} \text{meas} \{ \tau \in [0, T] : (p^{-i\tau} : p \in \mathbb{P}) \in A \}, \quad A \in \mathcal{B}(\Omega),
\]
converges weakly to the Haar measure \( m_H \) as \( T \to \infty \).

Proof. We use the method of Fourier transform and the linear independence over the field of rational numbers \( \mathbb{Q} \) for the set \( \{ \log p : p \in \mathbb{P} \} \). □

Lemma 2.5. The measure \( \hat{P}_{T,n} \) defined by
\[
\hat{P}_{T,n}(A) = \frac{1}{T} \text{meas} \{ \tau \in [0, T] : \zeta_n(s + i\tau, F) \in A \}, \quad A \in \mathcal{B}(H(D)),
\]
converges weakly to \( \hat{P}_n \) as \( T \to \infty \), where \( \hat{P}_n \) is defined in Lemma 2.3.

Proof. We use Lemma 2.4 and repeat the proof of Lemma 2.3. □

Lemma 2.6. \( \hat{P}_T \) converges weakly to \( P_\zeta \), and the support of \( P_\zeta \) is the set \( S \). Moreover, \( P_\zeta \) coincides with the limit measure \( P \) of \( \hat{P}_n \) as \( n \to \infty \).

Proof. We apply Lemma 2.5, the approximation of \( \zeta(s, F) \) and \( \zeta(s, \omega, F) \) by \( \zeta_n(s, F) \) and \( \zeta_n(s, \omega, F) \), respectively, and the classical Birkhoff-Khintchine ergodic theorem. For the investigation of the support, the positive density method is applied, see [7]. □

Our next aim is to show that the measure \( P_N \), as \( N \to \infty \), also converges weakly to the limit measure \( P \) of \( \hat{P}_n \) as \( n \to \infty \), i.e., that \( P_N \) converges weakly to \( P_\zeta \).

First we need a discrete version of approximation \( \zeta(s, F) \) by \( \zeta_n(s, F) \). Let \( \{K_l : l \in \mathbb{N}\} \subset D \) be a sequence of compact subsets such that \( D = \bigcup_{l=1}^{\infty} K_l \), \( K_l \subset K_{l+1} \) for all \( l \in \mathbb{N} \), and if \( K \subset D \) is a compact subset, then \( K \subset K_l \) for some \( l \in \mathbb{N} \). For \( g_1, g_2 \in H(D) \), set
\[
\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}.
\]
Then \( \rho \) is a metric on \( H(D) \) which induces its topology of uniform convergence on compacta.

We also recall the Gallagher lemma which relates continuous and discrete mean values of certain functions.

Lemma 2.7. Let \( T_0 \) and \( T \geq \delta > 0 \) be real numbers, and \( T \) be a finite set in the interval \([T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]\). Define
\[
N_\delta(x) = \sum_{t \in T} \mathbf{1}_{|t-x|<\delta},
\]
and let $S(x)$ be a complex-valued continuous function on $[T_0, T + T_0]$ having a continuous derivative on $(T_0, T + T_0)$. Then

$$\sum_{t \in T} N^{-1}(t)|S(t)|^2 \leq \frac{1}{\delta} \int_{T_0}^{T_0+T} |S(x)|^2 dx$$

$$+ \left( \int_{T_0}^{T_0+T} |S(x)|^2 dx \int_{T_0}^{T_0+T} |S'(x)|^2 dx \right)^{\frac{1}{2}}.$$

Proof of the lemma can be found in [13], Lemma 1.4.

Lemma 2.8. Suppose that $\alpha \in (0, 1)$ and $h > 0$ are fixed numbers. Then

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho(\zeta(s+ik^\alpha h, F), \zeta(s+ik^\alpha h, F)) = 0.$$

Proof. It is known that, for fixed $\sigma \in (\frac{\omega}{2}, \frac{\omega+1}{2})$,

$$\int_{0}^{T} |\zeta(\sigma + it, F)|^2 dt = O(T).$$

This together with the Cauchy integral formula implies, for the same $\sigma$, the estimate

$$\int_{0}^{T} |\zeta'(\sigma + it, F)|^2 dt = O(T).$$

Further, we will apply Lemma 2.7. For $2 \leq k \leq N$ and sufficiently large $N$, we have that

$$(k+1)^\alpha - k^\alpha = k^\alpha \left(1 + \frac{1}{k}\right)^\alpha - k^\alpha = k^\alpha \left(1 + \frac{\alpha}{k} + \frac{\alpha(\alpha-1)}{2k^2} + \cdots \right) - k^\alpha$$

$$= \frac{\alpha}{k^{1-\alpha}} + \frac{\alpha(\alpha-1)}{2k^{2-\alpha}} + \cdots > \frac{\alpha}{2N^{1-\alpha}}.$$

We take $\delta = \frac{\alpha h}{2N^{1-\alpha}}$ in Lemma 2.7. Then estimates (2.4), (2.5) and Lemma 2.7, for $\sigma \in (\frac{\omega}{2}, \frac{\omega+1}{2})$, yield

$$\sum_{k=0}^{N} |\zeta(\sigma + ik^\alpha h, F)|^2 \ll N^{1-\alpha} \int_{0}^{N^\alpha h} |\zeta(\sigma + it, F)|^2 dt$$

$$+ \left( \int_{0}^{N^\alpha h} |\zeta(\sigma + it, F)|^2 dt \right)^{\frac{1}{2}} \left( \int_{0}^{N^\alpha h} |\zeta'(\sigma + it, F)|^2 dt \right)^{\frac{1}{2}} \ll N.$$

Let $K$ be a compact subset of the strip $D$. Then, using (2.6) and contour integration, we find similarly to the proof of Theorem 4.1 from [5] that

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s \in K} (\zeta(s+ik^\alpha h, F) - \zeta(s+ik^\alpha h, F)) = 0.$$

This and the definition of the metric $\rho$ prove the lemma. □
Proof of Theorem 2.1. In view of Lemma 2.6, it suffices to show that \( P_N \) converges weakly to \( P \) as \( N \to \infty \), where \( P \) is the limit measure of \( \hat{P}_n \) as \( n \to \infty \).

Let \( \theta_N \) be a random variable defined on a certain probability space \((\Omega_0, \mathcal{A}, \mu)\), and having the distribution
\[
\mu(\theta_N = k\alpha h) = \frac{1}{N+1}, \quad k = 0, 1, \ldots, N.
\]
Define the \( H(D) \)-valued random element \( X_{N,n} \) by
\[
X_{N,n} = \zeta_n(s + i\theta_N, F).
\]
Then, by Lemmas 2.1 and 2.3, we have that \( X_{N,n} \) converges in distribution to \( \hat{X}_n \)
\[
\overset{D}{\longrightarrow}_{N \to \infty} \hat{X}_n, \tag{2.7}
\]
where \( \hat{X}_n \) is the \( H(D) \)-valued random element with the distribution \( \hat{P}_n \), and \( \hat{P}_n \) is the limit measure in Lemma 2.3. Since the series for \( \zeta_n(s, F) \) is absolutely convergent for \( \sigma > \frac{\kappa}{2} \), by a standard method it is easy to show that the family of probability measures \( \{\hat{P}_n : n \in \mathbb{N}\} \) is tight, i.e., for every \( \varepsilon > 0 \), there exists a compact subset \( K = K_F(\varepsilon) \subset D \) such that \( \hat{P}_n(K) > 1 - \varepsilon \) for all \( n \in \mathbb{N} \). Hence, by the Prokhorov theorem, see Theorem 6.1 in [1], the family \( \{\hat{P}_n\} \) is relatively compact. Thus, there exists a sequence \( \{\hat{P}_{n_r}\} \subset \{\hat{P}_n\} \) such that \( \hat{P}_{n_r} \) converges weakly to a certain probability measure \( \hat{P} \) on \((H(D), \mathcal{B}(H(D)))\) as \( r \to \infty \), i.e.,
\[
\overset{D}{\longrightarrow}_{r \to \infty} \hat{P}.
\]
(2.8)

On \((\Omega_0, \mathcal{A}, \mu)\), define one more \( H(D) \)-valued random element
\[
X_N = X_N(s) = \zeta(s + i\theta_N, F).
\]
Then, by Lemma 2.8, we find that, for every \( \varepsilon > 0 \),
\[
\lim_{n \to \infty} \lim_{N \to \infty} \mu(\rho(X_N, X_{N,n}) \geq \varepsilon) = \lim_{n \to \infty} \lim_{N \to \infty} \frac{1}{N+1} \#\{0 \leq k \leq N : \rho(\zeta_n(s + ik\alpha h, F), \zeta_n(s + ik\alpha h, F)) \geq \varepsilon\}
\leq \lim_{n \to \infty} \lim_{N \to \infty} \frac{1}{(N+1)\varepsilon} \sum_{k=0}^{N} \rho(\zeta_n(s + ik\alpha h, F), \zeta_n(s + ik\alpha h, F)) = 0.
\]
This and relations (2.7) and (2.8) show that all hypotheses of Theorem 4.2 of [1] are satisfied. Therefore,
\[
X_N \overset{D}{\longrightarrow}_{N \to \infty} \hat{P},
\]
or equivalently, \( P_N \) converges weakly to \( \hat{P} \) as \( N \to \infty \). Moreover, the latter relation shows that the measure \( \hat{P} \) is independent of the sequence \( \{\hat{P}_{n_r}\} \). Therefore,
\[
\hat{X}_n \overset{D}{\longrightarrow}_{n \to \infty} \hat{P},
\]
i.e., \( \hat{P}_n \) converges weakly to \( \hat{P} \) as \( n \to \infty \), thus \( \hat{P} = P \). Thus, we obtain that \( P_N \) converges weakly to the limit measure \( P \) of \( \hat{P}_n \) as \( n \to \infty \), and by Lemma 2.31 \( P \) coincides with \( P_\zeta \). The theorem is proved.

\[ \square \]

3. Proof of universality theorems

Proof of Theorem 1.3. By the Mergelyan theorem on the approximation of analytic functions by polynomials, there exists a polynomial \( p(s) \) such that

\[
\sup_{s \in K} |f(s) - e^{p(s)}| < \varepsilon/2.
\]

(3.1)

Define the set

\[ G = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - e^{p(s)}| < \varepsilon/2 \right\}. \]

Then, by Theorem 2.1, \( G \) is an open neighbourhood of the element \( e^{p(s)} \) of the support of the measure \( P_\zeta \). Hence, \( P_\zeta(G) > 0 \). This, Theorem 2.1 and an equivalent of the weak convergence of probability measures in terms of open sets show that

\[
\lim \inf_{N \to \infty} \frac{1}{N+1} \left\{ 0 \leq k \leq N : \zeta(s + ik\alpha h, F) \in G \right\} \geq P_\zeta(G) > 0,
\]

or, by the definition of \( G \), we have that

\[
\lim \inf_{N \to \infty} \frac{1}{N+1} \left\{ 0 \leq k \leq N : \sup_{s \in K} |\zeta(s + ik\alpha h, F) - e^{p(s)}| < \varepsilon/2 \right\} > 0.
\]

Combining this with (3.1) proves the theorem. \( \square \)

Proof of Theorem 1.4. It follows from Theorem 2.1, the continuity of the operator \( \Phi \) and Theorem 5.1 of [1] that the measure

\[
\frac{1}{N+1} \left\{ 0 \leq k \leq N : \Phi(\zeta(s + ik\alpha h, F)) \in A \right\}, \quad A \in \mathcal{B}(H(D)),
\]

converges weakly to \( P_\zeta \Phi^{-1} \) as \( N \to \infty \). Moreover, repeating the proof of Lemma 17 from [10], we obtain that the support of \( P_\zeta \Phi^{-1} \) includes the closure of the set \( H_{\Phi, a_1, ..., a_r}(D) \).

First suppose that \( f(s) \in H_{\Phi, a_1, ..., a_r}(D) \). Then, by the above remark, \( f(s) \) is an element of the support of \( P_\zeta \Phi^{-1} \). Therefore, putting

\[ G_1 = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}, \]

we have that \( P_\zeta \Phi^{-1}(G_1) > 0 \). This and the weak convergence of measure (3.2) prove the theorem in this case.

Now let \( r = 1 \). Then, by the Mergelyan theorem, there exists a polynomial \( p(s) \) such that

\[
\sup_{s \in K} |f(s) - p(s)| < \varepsilon/4.
\]

(3.3)
Since \( f(s) \neq a_1 \) on \( K \), by the Mergelyan theorem again, we can find a polynomial \( q(s) \) such that

\[
\sup_{s \in K} |p(s) - f_1(s)| < \varepsilon/4,
\]

(3.4)

where \( f_1(s) = a_1 + e^{q(s)} \). By the above remark, \( f_1(s) \) is an element of the support of the measure \( P_\xi \Phi^{-1} \). Therefore, if

\[
G_2 = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f_1(s)| < \varepsilon/2 \right\},
\]

then \( P_\xi \Phi^{-1}(G_2) > 0 \). Therefore, by the weak convergence of (3.2) to \( P_\xi \Phi^{-1} \) as \( N \to \infty \), we find that

\[
\lim \inf_{N \to \infty} \frac{1}{N + 1} \# \left\{ 0 \leq k \leq N : \sup_{s \in K} |\Psi(\zeta(s + ik^\alpha h, F)) - f_1(s)| < \varepsilon/2 \right\} > 0.
\]

This together with (3.3) and (3.4) prove the theorem in the case \( r = 1 \). □

References