DECOMPOSITIONS OF 2 × 2 MATRICES
OVER LOCAL RINGS

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Abstract. An element $a$ of a ring $R$ is called perfectly clean if there exists an idempotent $e \in \text{comm}^2(a)$ such that $a - e \in U(R)$. A ring $R$ is perfectly clean in case every element in $R$ is perfectly clean. In this paper, we completely determine when every $2 \times 2$ matrix and triangular matrix over local rings are perfectly clean. These give more explicit characterizations of strongly clean matrices over local rings. We also obtain several criteria for a triangular matrix to be perfectly J-clean. For instance, it is proved that for a commutative local ring $R$, every triangular matrix is perfectly J-clean in $T_n(R)$ if and only if $R$ is strongly J-clean.

1. Introduction

The commutant and double commutant of an element $a$ in a ring $R$ are defined by $\text{comm}(a) = \{x \in R \mid xa = ax\}$, $\text{comm}^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in \text{comm}(a)\}$, respectively. An element $a \in R$ is strongly clean provided that there exists an idempotent $e \in \text{comm}(a)$ such that $a - e \in U(R)$. A ring $R$ is called strongly clean in the case that every element in $R$ is strongly clean. Strongly clean matrix rings and triangular matrix rings over local rings have been extensively studied by many authors (cf. [1, 2, 5, 6] and [12, 13]). An element $a \in R$ is quasipolar provided that there exists an idempotent $e \in \text{comm}(a)$ such that $a + e \in U(R)$ and $ae \in R^{\text{qnil}}$, where $R^{\text{qnil}} = \{x \in R \mid 1 + xr \in U(R) \text{ for any } r \in \text{comm}(x)\}$. A ring $R$ is called quasipolar if every element in $R$ is quasipolar. As is well known, a ring $R$ is quasipolar if and only if for any $a \in R$ there exists a $b \in \text{comm}^2(a)$ such that $b = bab$ and $b - b^2a \in R^{\text{qnil}}$. This concept has evolved from Banach algebra. In fact, for a Banach algebra $R$, $a \in R^{\text{qnil}} \iff \lim_{n \to \infty} \|a^n\|^\frac{1}{n} = 0$.

It is shown that every quasipolar ring is strongly clean. Recently, quasipolar $2 \times 2$ matrix rings and triangular matrix rings over local rings were also studied from different point of views (cf. [7, 9, 11]).
The motivation for this article is to introduce a medium class between strongly clean rings and quasipolar rings, and then explore more explicit decompositions of $2 \times 2$ matrices over a local ring. An element $a$ of a ring $R$ is called perfectly clean if there exists an idempotent $e \in \text{comm}^2(a)$ such that $a - e \in U(R)$. A ring $R$ is perfectly clean in the case every element in $R$ is perfectly clean. We completely determine when every $2 \times 2$ matrix and triangular matrix over local rings are perfectly clean. These also give more explicit characterizations of strong clean matrices over local rings, and enhance many known results, e.g., [5, Theorem 8], [11, Theorem 2.8] and [12, Theorem 7]. Replaced $U(R)$ by $J(R)$, we introduce perfectly $J$-clean rings as a subclass of perfectly clean rings. Furthermore, we show that strong $J$-cleanness for triangular matrices over a local ring can be enhanced to such stronger properties. These also generalize the corresponding properties of J-quasipolarity, e.g., [8, Theorem 4.9].

We write $U(R)$ and $J(R)$ for the set of all invertible elements and the Jacobson radical of $R$; $M_n(R)$ and $T_n(R)$ stand for the rings of all $n \times n$ matrices and triangular matrices over a ring $R$.

2. Perfect rings

Clearly, an abelian exchange ring is perfectly clean. Every quasipolar ring is perfectly clean. For instance, every strongly $\pi$-regular ring. In fact, we have $\{\text{quasipolar rings}\} \subsetneq \{\text{perfectly clean rings}\} \subsetneq \{\text{strongly clean rings}\}$. In this section, we explore the properties of perfect rings, which will be used in the sequel. We begin with

Theorem 2.1. Let $R$ be a ring. Then the following are equivalent:

1. $R$ is perfectly clean.
2. For any $a \in R$, there exists an $x \in \text{comm}^2(a)$ such that $x = xax$ and $1 - x \in (1 - a)R \cap R(1 - a)$.

Proof. (1) $\Rightarrow$ (2) For any $a \in R$, there exists an idempotent $e \in \text{comm}^2(a)$ such that $u := a - e \in U(R)$. Set $x = u^{-1}(1 - e)$. Let $y \in \text{comm}(a)$. Then $ay = ya$. As $uy = (a - e)y = y(a - e) = yu$, we get $u^{-1}y = yu^{-1}$. Thus, $xy = u^{-1}(1 - e)y = u^{-1}y(1 - e) = yu^{-1}(1 - e) = xy$. This implies that $x \in \text{comm}^2(a)$. Further, $xax = u^{-1}(1 - e)(u + e)u^{-1}(1 - e) = u^{-1}(1 - e) = x$. Clearly, $u = (1 - e) - (1 - a)$, and so $1 - u^{-1}(1 - e) = u^{-1}(1 - a)$. This implies that $1 - u \in R(1 - a)$. Likewise, $1 - x \in (1 - a)R$ as $(1 - e)u^{-1} = u^{-1}(1 - e)$. Therefore $1 - x \in (1 - a)R \cap R(1 - a)$, as required.

(2) $\Rightarrow$ (1) For any $a \in R$, there exists an $x \in \text{comm}^2(a)$ such that $x = xax$ and $1 - x \in (1 - a)R \cap R(1 - a)$. Write $e = 1 - ax$. If $y \in \text{comm}(a)$, then $ay = ya$, and so $axy = ayx$. This shows that $e(y = ye$; hence, $e \in \text{comm}^2(a)$. In addition, $ex = xe = 0$. Write $1 - x = (1 - a)s = t(1 - a)$ for some $s, t \in R$. Then

$$(a - e)(x - es) = ax - aes + es = ax + (1 - a)es$$

$$= ax + e(1 - a)s = ax + e(1 - x) = ax + e = 1.$$ 

Likewise, $(x - te)(a - e) = 1$. Therefore $a - e \in U(R)$, as desired. $\square$
Corollary 2.1. Let $R$ be a ring. Then the following are equivalent:

1. $R$ is perfectly clean.

2. For any $a \in R$, there exists an idempotent $e \in \text{comm}^2(a)$ such that $eae \in U(eRe)$ and $(1-e)(1-a)(1-e) \in U((1-e)R(1-e))$.

Proof. (1) $\Rightarrow$ (2) For any $a \in R$, it follows from Theorem 2.1 that there exists an $x \in \text{comm}^2(a)$ such that $x = xax$ and $1-x \in (1-a)R \cap R(1-a)$. Write $1-x = (1-a)s = t(1-a)$ for some $s, t \in R$. Set $e = ax$. For any $y \in \text{comm}(a)$, we have $ay = ya$, and so $ey = (ax)y = a(yx) = (ay)x = y(ax) = ye$. Hence, $e^2 = e \in \text{comm}^2(a)$. Clearly, $(eae)(exe) = (exe)(eae) = e$; hence, $eae \in U(eRe)$. Furthermore, $1-e = (1-x)+(1-a)x = (1-a)(s+x)$. This shows that $(1-e)(1-a)(1-x)(1-e) = 1-e$. Likewise, $(1-e)(1-x)(1-e)(1-a)(1-x) = 1-e$. Therefore $(1-e)(1-a)(1-e) \in U((1-e)R(1-e))$.

(2) $\Rightarrow$ (1) For any $a \in R$, we have an idempotent $e \in \text{comm}^2(a)$ such that $eae \in U(eRe)$ and $(1-e)(1-a)(1-e) \in U((1-e)R(1-e))$. Hence, $a - (1-e) = (eae - (1-e)(1-a)(1-e)) \in U(R)$. Set $p = 1-e$. Then $a-p \in U(R)$ with $p \in \text{comm}^2(a)$, as desired. □

Recall that a ring $R$ is strongly nil clean provide that every element in $R$ is the sum of an idempotent and a nilpotent element that commute (cf. [4] and [10]).

Theorem 2.2. Let $R$ be a ring. Then $R$ is strongly nil clean if and only if

1. $R$ is perfectly clean, 2. $N(R) = \{ x \in R \mid 1-x \in U(R) \}$.

Proof. Let $R$ be strongly nil clean. For any $a \in R$, we see that $a - a^2 \in N(R)$. Write $(a - a^2)^n = 0$. Let $f(t) = \sum_{i=0}^{n} \binom{2n}{i} t^{2n-i}(1-t)^i \in \mathbb{Z}[t]$. Then we have $f(t) \equiv 1 \pmod{(1-t)^n}$. Clearly, $f(t) + \sum_{i=n+1}^{2n} \binom{2n}{i} t^{2n-i}(1-t)^i = (t + (1-t))^n = 1$; hence, $f(t) \equiv 1 \pmod{(1-t)^n}$. This shows that $f(t)(1-f(t)) \equiv 0 \pmod{(1-t)^n}$. Let $e = f(a)$. Then $e \in R$ is an idempotent. For any $x \in \text{comm}(a)$, we see that $xa = ax$, and so $xe = xf(a) = f(a)x = ex$. Thus, $e \in \text{comm}^2(a)$. Furthermore, $a - e = a - a^2 + (a - 2a^2)g(a) = (a - a^2)(1+a^2 + \cdots + a^{2n-2}) + g(a) \in N(R)$, where $g(t) \in \mathbb{Z}[t]$. Thus, $a = (1-e) + (2e - 1 + a - e)$ with $1-e \in \text{comm}^2(a)$ and $2e - 1 + a - e \in U(R)$. Therefore, $R$ is perfectly clean.

Clearly, $N(R) \subseteq \{ x \in R \mid 1-x \in U(R) \}$. If $1-x \in U(R)$, then $x = e+w$ with $e \in \text{comm}(x)$ and $w \in N(R)$. Hence, $1-e = (1-x) + w \in U(R)$. This implies that $1-e = 1$, and so $x = w \in N(R)$. Therefore $N(R) = \{ x \in R \mid 1-x \in U(R) \}$.

Conversely, assume that (1) and (2) hold. For any $a \in R$, there exist an idempotent $e \in \text{comm}^2(a)$ and a unit $u \in R$ such that $-a = e - u$. Hence, $a = e + u = (1-e) - (1-u)$. By hypothesis, $1-u \in N(R)$. Accordingly, $R$ is strongly nil clean. □

Corollary 2.2. Let $R$ be a ring. Then $R$ is strongly nil clean if and only if

1. $R$ is quasipolar; 2. $N(R) = \{ x \in R \mid 1-x \in U(R) \}$.
Proof. Suppose that \( R \) is strongly nil clean. Then (2) holds by Theorem 2.2. For any \( a \in R \), as in the proof of Theorem 2.2, \( a = e + w \) with \( e \in \mathrm{comm}^2(a) \) and \( w \in N(R) \). Hence, \( a = (1 - e) + (2e - 1 + w) \) where \( 2e - 1 + w \in U(R) \). Furthermore, \( (1 - e)a = (1 - e)w \in N(R) \subseteq R^{\mathrm{nil}} \). Therefore \( R \) is quasipolar.

Conversely, assume that (1) and (2) hold. Then \( R \) is perfectly clean. Accordingly, we complete the proof by Theorem 2.2.

Lemma 2.1. Let \( R \) be a ring. Then the following are equivalent:

1. \( R \) is perfectly clean.
2. For each \( a \in R \) there exists an idempotent \( e \in \mathrm{comm}^2(a) \) such that \( a - e \) and \( a + e \) are invertible.

Proof. (1) \( \Rightarrow \) (2) Let \( a \in R \). Then \( a^2 \in R \) is perfectly clean. Thus, we can find an idempotent \( e \in \mathrm{comm}^2(a^2) \) such that \( a^2 - e \in U(R) \). Since \( a \cdot a^2 = a^2 \cdot a \), we see that \( ae = ea \). Hence, \( a^2 - e = (a - e)(a + e) \), and therefore we conclude that \( a - e, a + e \in U(R) \).

(2) \( \Rightarrow \) (1) is trivial.

Theorem 2.3. Let \( R \) be perfectly clean. Then for any \( A \in M_n(R) \) there exist \( U, V \in \mathrm{GL}_n(R) \) such that \( 2A = U + V \).

Proof. We prove the result by induction on \( n \). For any \( a \in R \), there exists an idempotent \( e \in \mathrm{comm}^2(a) \) such that \( u := a - e, v := a + e \in U(R) \), by Lemma 2.1. Hence, \( 2a = u + v \), and so the result holds for \( n = 1 \). Assume that the result holds for \( n \leq k \) (\( k \geq 1 \)). Let \( n = k + 1 \), and let \( A \in M_n(R) \). Write \( A = \begin{pmatrix} \alpha & \beta \\ \beta & X \end{pmatrix} \), where \( x \in R, \alpha \in M_{1 \times k}(R), \beta \in M_{k \times 1}(R) \) and \( X \in M_k(R) \). In view of Lemma 2.1, we have a \( u \in U(R) \) such that \( 2x - u = v \in U(R) \). By hypothesis, we have a \( U \in \mathrm{GL}_k(R) \) such that \( 2(X - 2\beta v^{-1} \alpha) - U = V \in \mathrm{GL}_k(R) \). Hence

\[
2A - \begin{pmatrix} u & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} v & 2\alpha \\ 2\beta & V + 4\beta v^{-1} \alpha \end{pmatrix}.
\]

It is easy to verify that

\[
\begin{pmatrix} v & 2\alpha \\ 2\beta & V + 4\beta v^{-1} \alpha \end{pmatrix} = \begin{pmatrix} 1 & 2\beta v^{-1} \\ 2\beta v^{-1} I_k \end{pmatrix} \begin{pmatrix} v & 2\alpha \\ 0 & V \end{pmatrix} \in \mathrm{GL}_n(R).
\]

By induction, we complete the proof.

Corollary 2.3. Let \( R \) be a quasipolar ring. If \( \frac{1}{2} \in R \), then every \( n \times n \) matrix over \( R \) is the sum of two invertible matrices.

Proof. As every quasipolar ring is perfectly clean, the proof follows by Theorem 2.3.

As a consequence, we derive the following known fact: Let \( R \) be a strongly \( \pi \)-regular ring with \( \frac{1}{2} \in R \). Then every \( n \times n \) matrix over \( R \) is the sum of two invertible matrices.
3. Matrices and triangular matrices

Recall that a ring $R$ is local if it has only one maximal right ideal. A ring $R$ is local if and only if for any $a \in R$ either $a$ or $1 - a$ is invertible. Necessary and sufficient conditions under which $2 \times 2$ matrices over a local ring are attractive. In this section, we extend these known results on strongly clean matrices to perfect cleanness.

**Lemma 3.1.** Let $R$ be a ring, and $u \in U(R)$. Then the following are equivalent:

1. $a \in R$ is perfectly clean.
2. $uau^{-1} \in R$ is perfectly clean.

**Proof.** (1) $\Rightarrow$ (2) By hypothesis, there exists an idempotent $e \in \text{comm}^2(a)$ such that $a - e \in U(R)$. Hence, $uau^{-1} - ueu^{-1} \in U(R)$. For any $x \in \text{comm}(uau^{-1})$, we see that $x(uau^{-1}) = (uau^{-1})x$, and so $(u^{-1}ux)a = a(u^{-1}ux)$. Thus, $(u^{-1}ux)e = e(u^{-1}ux)$. Hence $x(uau^{-1}) = (uau^{-1})x$. We conclude that $ueu^{-1} \in \text{comm}^2(uau^{-1})$, as required.

(2) $\Rightarrow$ (1) is symmetric. \qed

A ring is weakly cobleached provided that for any $a \in J(R)$, $b \in 1 + J(R)$, $l_a - r_b$ and $l_b - r_a$ are both injective. For instance, every commutative local ring, every local ring with nil Jacobson radical.

**Theorem 3.1.** Let $R$ be a weakly cobleached local ring. Then the following are equivalent:

1. $M_2(R)$ is perfectly clean.
2. $M_2(R)$ is strongly clean.
3. For any $A \in M_2(R)$, $A \in \text{GL}_2(R)$, or $I_2 - A \in \text{GL}_2(R)$,
or $A$ is similar to a diagonal matrix.

**Proof.** (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) is obtained by [13, Theorem 7].

(3) $\Rightarrow$ (1) For any $A \in M_2(R)$, $A \in \text{GL}_2(R)$, or $I_2 - A \in \text{GL}_2(R)$, or $A$ is similar to a diagonal matrix. If $A$ or $I_2 - A \in \text{GL}_2(R)$, then $A$ is perfectly clean. Assume now that $A$ is similar to a diagonal matrix with $A, I_2 - A \notin \text{GL}_2(R)$. We may assume that $A$ is similar to $(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix})$, where $\lambda \in U(R), \mu \in J(R)$. If $\lambda \in 1 + U(R)$, then $(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}) - I_2 \notin \text{GL}_2(R)$; hence, it is perfectly clean. In view of Lemma 3.1, $A$ is perfectly clean. Thus, we assume that $\lambda \in 1 + J(R)$. By Lemma 3.1, it will suffice to show that $(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}) \in \text{GL}_2(R)$ is perfectly clean. Clearly,

$$(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}) = (\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}) + (\begin{smallmatrix} \lambda & 0 \\ 0 & \mu - 1 \end{smallmatrix}),$$

where $(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}) \in \text{GL}_2(R)$.

We show that the idempotent $(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}) \in \text{comm}^2((\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}))$. For any $(\begin{smallmatrix} s & t \\ y & y \end{smallmatrix}) \in \text{comm}((\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}))$, one has $\lambda s = sy$ and $\mu t = t\lambda$; hence, $s = t = 0$. This implies

$$(\begin{smallmatrix} x & s \\ t & y \end{smallmatrix})(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}) = (\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}) = (\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix})(\begin{smallmatrix} x & s \\ t & y \end{smallmatrix}).$$

Therefore $(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}) \in \text{comm}^2((\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}))$, hence the result. \qed
COROLLARY 3.1. Let $R$ be a commutative local ring. Then the following are equivalent:

1. $M_2(R)$ is perfectly clean.
2. $M_2(R)$ is strongly clean.
3. For any $A \in M_2(R)$, $A \in \text{GL}_2(R)$, or $I_2 - A \in \text{GL}_2(R)$, or $A$ is similar to a diagonal matrix.

PROOF. It is a consequence of Theorem 3.1 as every commutative local ring is weakly cobleached. \qed 

Let $p$ be a prime. We use $\mathbb{Z}_p$ to denote the ring of all $p$-adic integers. In view of [6] Theorem 2.4, $M_2(\mathbb{Z}_p)$ is strongly clean, and therefore $M_2(\mathbb{Z}_p)$ is perfectly clean, by Corollary 3.1.

THEOREM 3.2. Let $R$ and $S$ be local rings. Then the following are equivalent:

1. \( \left( \frac{R}{S} \right) \) is perfectly clean.
2. For any $a \in J(R)$, $b \in 1 + J(S)$, $v \in V$, there exists a unique $x \in V$ such that $ax - xb = v$.

PROOF. (1) $\Rightarrow$ (2) Let $a \in 1 + J(R)$, $b \in J(S)$ and $v \in V$. Set $A = \begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix}$. By hypothesis, we can find an idempotent $E \in \text{comm}^2(A)$ such that $A - E \in \left( \frac{R}{S} \right)$ is invertible. Clearly, $E = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}$ for some $x \in V$. Thus, $ax - xb = v$. Suppose that $ay - xb = v$ for a $y \in V$. Then

\[
A \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix} A,
\]

and so $\begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix} \in \text{comm}(A)$. This implies that

\[
E \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix} E;
\]

hence, $x = y$. Therefore there exists a unique $x \in V$ such that $ax - xb = v$, as desired.

(2) $\Rightarrow$ (1) Let $T = \left( \frac{R}{S} \right)$, and let $A = \begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix} \in \left( \frac{R}{S} \right)$.

Case I. $a \in J(R), b \in J(S)$. Then $A - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in T$; hence, $A$ is perfectly clean.

Case II. $a \in U(R), b \in U(S)$. Then $A - 0 \in U(T)$; hence, $A$ is perfectly clean.

Case III. $a \in U(R), b \in J(S)$. (i) $a \in 1 + U(R), b \in J(S)$. Then $A - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in T$ is invertible; hence, $A \in T$ is perfectly clean. (ii) $a \in 1 + J(R), b \in J(S)$. Then we can find a $t \in V$ such that $at - tb = -v$. Let $\begin{pmatrix} x & s \\ 0 & y \end{pmatrix} \in \text{comm}(A)$. Then

\[
A \begin{pmatrix} x & s \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & s \\ 0 & y \end{pmatrix} A,
\]

and so $ax = xa$, $by = yb$, and $as - sb = xv - vy$. Hence, we check that

\[
a(xt - ty + s) - (xt - ty + s)b = x(at - tb) - (at - tb)y + (as - sb)
\]

\[= -xv + vy + (as - sb)
\]

\[= 0.
\]
By hypothesis, \( xt - ty = -s \), and so we get
\[
\begin{pmatrix}
0 & t \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x & s \\
0 & y
\end{pmatrix}
= \begin{pmatrix}
0 & ty \\
0 & y
\end{pmatrix}
= \begin{pmatrix}
x & s \\
0 & y
\end{pmatrix}
\begin{pmatrix}
0 & t \\
0 & 1
\end{pmatrix}.
\]
We infer that
\[
\begin{pmatrix}
0 & t \\
0 & 1
\end{pmatrix}^2
- \begin{pmatrix}
0 & t \\
0 & 1
\end{pmatrix} \in \text{comm}^2(A).
\]
Furthermore, \( A - \begin{pmatrix}
0 & t \\
0 & 1
\end{pmatrix} \in U(T) \). Therefore \( A \) is perfectly clean.

**Case IV.** \( a \in J(R), b \in U(S) \) Then \( A \) is perfectly clean, as in the preceding discussion.

A ring \( R \) is uniquely weakly bleached provided that for any \( a \in J(R), b \in 1 + J(R), l_a - r_b \) and \( l_b - r_a \) are both isomorphisms.

**Corollary 3.2.** Let \( R \) be local. Then the following are equivalent:

1. \( T_2(R) \) is perfectly clean.
2. \( R \) is uniquely weakly bleached.

**Proof.** It is clear by Theorem 3.2.

**4. Perfectly J-clean rings**

An element \( a \in R \) is said to be perfectly J-clean provided that there exists an idempotent \( e \in \text{comm}^2(a) \) such that \( a - e \in J(R) \). A ring \( R \) is perfectly J-clean if every element in \( R \) is perfectly J-clean.

**Theorem 4.1.** Let \( R \) be a ring. Then \( R \) is perfectly J-clean if and only if

1. \( R \) is quasipolar.
2. \( R/J(R) \) is Boolean.

**Proof.** Suppose that \( R \) is perfectly J-clean. Let \( a \in R \) is perfectly J-clean. Then there exists an idempotent \( e \in \text{comm}^2(a) \) such that \( w := a - e \in J(R) \). Hence, \( a - (1 - e) = 2e - 1 + w \in U(R) \). Additionally, \( (1 - e)a = (1 - e)w \in J(R) \subseteq R^\text{qnil} \).

This implies that \( a \in R \) is quasipolar. Furthermore, \( a - a^2 = (e + w) - (e + w)^2 \in J(R) \), and then \( R/J(R) \) is Boolean.

Conversely, assume that (1) and (2) hold. Let \( a \in R \). Then there exists an idempotent \( e \in \text{comm}^2(a) \) such that \( u := a - e \in U(R) \). Moreover, \( R/J(R) \) is Boolean, and so \( a - a^2 = (e + u) - (e + u)^2 \in (1 - 2e - u) \in J(R) \). This shows that \( 1 - 2e - u \in J(R) \), whence \( a - (1 - e) = (e + u) - (1 - e) = 2e - 1 + u \in J(R) \).

Therefore \( R \) is perfectly J-clean.

**Corollary 4.1.** Let \( R \) be a ring. Then the following are equivalent:

1. \( R \) is perfectly J-clean.
2. \( R \) is perfectly clean, and \( R/J(R) \) is Boolean.
3. \( R \) is quasipolar, and \( R \) is strongly J-clean.
Proof. (1) ⇒ (2) is obvious by Theorem 4.1 as every quasipolar ring is perfectly clean.

(2) ⇒ (1) For any \( a \in R \) there exists an idempotent \( p \in \text{comm}(a) \) such that \( u := a - p \in U(R) \). As \( R/J(R) \) is Boolean, we have \( u = u^2 \); hence, \( u \in 1 + J(R) \). Furthermore, \( 2 \in J(R) \). Accordingly, \( a = p + u = (1 - p) + (2p - 1 + u) \) with \( 1 - p \in \text{comm}(a) \) and \( 2p - 1 + u \in J(R) \), as desired.

(1) ⇒ (3) Suppose \( R \) is perfectly \( J \)-clean. Then \( R \) is strongly \( J \)-clean. By the preceding discussion, \( R \) is quasipolar.

(3) ⇒ (1) Since \( R \) is strongly \( J \)-clean, \( R/J(R) \) is Boolean. Therefore the proof is complete by the discussion above.

\[ \square \]

Example 4.1. Let \( R = T_2(Z_{2^n}) \) (\( n \in \mathbb{N} \)). Then \( T_2(R) \) is perfectly \( J \)-clean.

Proof. As \( R \) is finite, it is periodic. This shows that \( R \) is strongly \( \pi \)-regular. Hence, \( T_2(R) \) is quasipolar, by [9, Theorem 2.6]. As \( J(Z_{2^n}) = 2Z_{2^n} \), we see that \( R/J(R) \cong Z_2 \) is Boolean. Hence, \( T_2(R)/J(T_2(R)) \) is Boolean. Therefore the result follows by Theorem 4.1.

Recall that a ring \( R \) is uniquely strongly clean provided that for any \( a \in R \) there exists a unique idempotent \( e \in \text{comm}(a) \) such that \( a - e \in U(R) \).

Proposition 4.1. Let \( R \) be a ring. Then \( R \) is perfectly \( J \)-clean if and only if \( \begin{array}{ll}
(1) \ R \text{ is perfectly clean}, & (2) \ R \text{ is uniquely strongly clean}.
\end{array} \)

Proof. Suppose \( R \) is perfectly \( J \)-clean. Then \( R \) is perfectly clean. Hence, \( R \) is strongly clean. Let \( a \in R \). Write \( a = e + u = f + v \) with \( e = e^2 \in \text{comm}(a) \), \( f = f^2 \in R \), \( u \in J(R) \), \( v \in U(R) \), \( ea = ae \) and \( fa = af \). Then \( f \in \text{comm}(a) \), and so \( ef = fe \). Thus, \( e - f = v - u \in U(R) \) and \( (e - f)(e + f - 1) = 0 \). This implies that \( f = 1 - e \), and therefore \( R \) is uniquely strongly clean.

Conversely, assume that (1) and (2) hold. Then \( R/J(R) \) is Boolean. Therefore we complete the proof by Corollary 4.1.

Corollary 4.2. A ring \( R \) is uniquely clean if and only if \( R \) is abelian perfectly \( J \)-clean.

Proof. As every uniquely clean ring is abelian (cf. [4, Corollary 16.4.16]), it is clear by Proposition 4.1.

\[ \square \]

Theorem 4.2. Let \( R \) be a ring. Then the following are equivalent:

\( \begin{array}{ll}
(1) \ R \text{ is perfectly } J \text{-clean}, & (2) \text{ For any } a \in R, \text{ there exists a unique idempotent } e \in \text{comm}(a) \text{ such that } a - e \in J(R).
\end{array} \)

Proof. (1) ⇒ (2) For any \( a \in R \), there exists an idempotent \( e \in \text{comm}(a) \) such that \( a - e \in J(R) \). Assume that \( a - f \in J(R) \) for an idempotent \( f \in \text{comm}(a) \). Clearly, \( e \in \text{comm}(a) \subseteq \text{comm}(a) \). As \( f \in \text{comm}(a) \), we see that \( ef = fe \). Thus, \( (e - f)^2 = e - f \), and so \( (e - f)(1 - (e - f)^2) = 0 \). But \( e - f = (a - f) - (a - e) \in J(R) \), as \( a - f, a - e \in J(R) \). Hence, \( e = f \), as desired.

(2) ⇒ (1) is trivial.

\[ \square \]
Recall that a ring $R$ is strongly $J$-clean provided that for any $a \in R$ there exists an idempotent $e \in \text{comm}(a)$ such that $a - e \in J(R)$ (cf. [3, 4]).

**Corollary 4.3.** A ring $R$ is perfectly $J$-clean if and only if

1. $R$ is quasipolar,
2. $R$ is strongly $J$-clean.

**Proof.** Suppose $R$ is perfectly $J$-clean. Then $R$ is strongly $J$-clean. For any $a \in R$, there exists an idempotent $p \in \text{comm}^2(a)$ such that $w := a - p \in J(R)$. Hence, $a = (1 - p) + (2p - 1 + w)$ with $1 - p \in \text{comm}^2(a)$ and $2p - 1 + w \in U(R)$. Furthermore, $(1 - p)a = (1 - p)w \in J(R) \subseteq R^\text{qnil}$. Therefore, $R$ is quasipolar.

Conversely, assume that (1) and (2) hold. Since $R$ is quasipolar, it is perfectly clean. By virtue of [4 Proposition 16.4.15], $R/J(R)$ is Boolean. Therefore the proof is complete by Corollary 4.1.

Following Cui and Chen [8], a ring $R$ is called $J$-quasipolar provided that for any element $a \in R$ there exists an $e \in \text{comm}^2(a)$ such that $a + e \in J(R)$. We further show that the two concepts coincide. But this is not the case for a single element. That is,

**Proposition 4.2.** A ring $R$ is perfectly $J$-clean if and only if for any element $a \in R$ there exists an $e \in \text{comm}^2(a)$ such that $a + e \in J(R)$.

**Proof.** Let $R$ be perfectly $J$-clean. Then $R/J(R)$ is Boolean, by Theorem 4.1. Hence, $2^2 = 2$, i.e., $2 \in J(R)$. For any $a \in R$, there exists an idempotent $e \in \text{comm}^2(a)$ such that $a - e \in J(R)$. This implies that $a + e = (a - e) + 2e \in J(R)$. The converse is similar by [8 Corollary 2.3].

**Example 4.2.** Let $R = \mathbb{Z}_3$. Note that $J(R) = 0$. Since $\bar{1} - \bar{1} = \bar{0} \in J(R)$, $\bar{1}$ is perfectly $J$-clean, but we cannot find an idempotent $e \in R$ such that $\bar{1} + e \in J(R)$, because $\bar{1} + 0 \notin J(R)$ and $\bar{1} + 1 = 2 \notin J(R)$.

Further, though $\bar{2} + \bar{1} = \bar{0} \in J(R)$, we cannot find an idempotent $e \in R$ such that $\bar{2} + e \in J(R)$, because $\bar{2} + \bar{0} = 2 \notin J(R)$ and $\bar{2} - 1 = \bar{1} \notin J(R)$.

**Lemma 4.1.** Let $R$ be a ring. Then $a \in R$ is perfectly $J$-clean if and only if

1. $a \in R$ is quasipolar,
2. $a - a^2 \in J(R)$.

**Proof.** Suppose that $a \in R$ is perfectly $J$-clean. Then there exists an idempotent $e \in \text{comm}^2(a)$ such that $w := a - e \in J(R)$. Hence, $a - (1 - e) = 2e - 1 + w \in U(R)$. Additionally, $(1 - e)a = (1 - e)w \in J(R) \subseteq R^\text{qnil}$. This implies that $a \in R$ is quasipolar. Furthermore, $(e + w) - (e + w)^2 = -(2e - 1 + w)w \in J(R)$.

Conversely, assume that (1) and (2) hold. Then there exists an idempotent $e \in \text{comm}^2(-a)$ such that $(-a) + e \in U(R)$. Set $u := a - e$. Then $a - a^2 = (e + u) - (e + u)^2 = u(1 - 2e - u) \in J(R)$; hence, $1 - 2e - u \in J(R)$. This shows that $a - (1 - e) = (e + u) - (1 - e) = 2e - 1 + u \in J(R)$. Therefore $a \in R$ is perfectly $J$-clean.

**Theorem 4.3.** Let $R$ be a commutative ring, and let $A \in T_n(R)$. If $2 \in J(R)$, then the following are equivalent:

1. $A \in T_n(R)$ is perfectly $J$-clean.
2. Each $A_{ii} \in T_n(R)$ is perfectly $J$-clean.
E \beta

Furthermore, \( E_1 \in \text{comm}^2(A_1) \), we get
\[
\gamma(A_1 - a_{11}I_n - 1)(E_1 - e_{11}I_{n-1}) = \gamma(E_1 - e_{11}I_{n-1})(E_1 + W_1 - (e_{11} + w_{11})I_{n-1}) = \gamma(E_1 - e_{11}I_{n-1})(E_1 + e_{11}I_{n-1} + (W_1 - 2e_{11} - w_{11})I_{n-1}) = \gamma(E_1 - e_{11}I_{n-1})(E_1 - e_{11}I_{n-1})(W_1 + (1 - 2e_{11} - w_{11})I_{n-1}).
\]

It follows from \( W_1 + (1 - 2e_{11} - w_{11})I_{n-1} \in U(T_{n-1}(R)) \) that \( \gamma(E_1 - e_{11}I_{n-1}) = \beta(X_1 - x_{11}I_{n-1}) \). Hence, \( e_{11}X_1 = x_{11}\beta + \gamma E_1 \), and so \( EX = XE \). This implies that \( E \in \text{comm}^2(A) \). By induction, \( A \in T_n(R) \) is perfectly J-clean for all \( n \in \mathbb{N} \).

**Corollary 4.4.** Let \( R \) be a commutative ring. Then the following are equivalent:

1. \( R \) is strongly J-clean.
2. \( T_n(R) \) is perfectly J-clean for all \( n \in \mathbb{N} \).
3. \( T_n(R) \) is perfectly J-clean for some \( n \in \mathbb{N} \).
Proof. (1) ⇒ (2) As $R$ is strongly J-clean, $R/J(R)$ is Boolean. Hence, $2 \in J(R)$. For any $n \in \mathbb{N}$, $T_n(R)$ is perfectly J-clean by Theorem 4.3.

(2) ⇒ (3) ⇒ (1) These are clear by Theorem 4.3. □

Let $R$ be Boolean. As a consequence of Corollary 4.4, $T_n(R)$ is perfectly J-clean for all $n \in \mathbb{N}$.

Lemma 4.2. Let $R$ be a ring, and $u \in U(R)$. Then the following are equivalent:

(1) $a \in R$ is perfectly J-clean.

(2) $uau^{-1} \in R$ is perfectly J-clean.

Proof. (1) ⇒ (2) As in the proof of Lemma 3.1, $uau^{-1} \in R$ is quasipolar. Furthermore, $uau^{-1} - (uau^{-1})^2 = u(a - a^2)u^{-1} \in J(R)$. As in the proof of Theorem 4.1, $uau^{-1} \in R$ is perfectly J-clean.

(2) ⇒ (1) is symmetric. □

We end this paper by showing that strong J-cleanness of $2 \times 2$ matrix ring over a commutative local ring can be enhanced to perfect J-cleanness.

Theorem 4.4. Let $R$ be a commutative local ring, and let $A \in M_2(R)$. Then the following are equivalent:

(1) $A$ is perfectly J-clean.

(2) $A$ is strongly J-clean.

(3) $A \in J(M_2(R))$, or $I_2 - A \in J(M_2(R))$, or the equation $x^2 - tr(A)x + \det(A) = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.

Proof. (1) ⇒ (2) is trivial.

(2) ⇒ (3) is proved by [4, Theorem 16.4.31].

(3) ⇒ (1) If $A \in J(M_2(R))$ or $I_2 - A \in J(M_2(R))$, then $A$ is perfectly J-clean.

Otherwise, it follows from [4, Theorem 16.4.31 and Proposition 16.4.26] that there exists a $U \in GL_2(R)$ such that

$$U AU^{-1} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta - 1 \end{pmatrix},$$

where $\alpha \in J(R), \beta \in 1 + J(R)$. For any $X \in \text{comm}(U AU^{-1})$, we have $X \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} X$; hence, $\beta X_{12} = \alpha X_{12}$. This implies that $X_{12} = 0$. Likewise, $X_{21} = 0$. Thus,

$$X \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} X,$$

and so $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \text{comm}(U AU^{-1})$. As a result, $U AU^{-1}$ is perfectly J-clean, and then so is $A$ by Lemma 4.2. □

Corollary 4.5. Let $R$ be a commutative local ring. Then the following are equivalent:

(1) $M_2(R)$ is perfectly clean.

(2) For any $A \in M_2(R)$, $A \in GL_2(R)$, or $I_2 - A \in GL_2(R)$, or $A \in M_2(R)$ is perfectly J-clean.

Proof. (1) ⇒ (2) is proved by Theorem 4.4. □

(2) ⇒ (1) is obvious. □
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