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USES OF THE LANGUAGE OF MATHEMATICS

SUMMARY: In this paper I criticise the dogma that asserting and naming are the most important language uses in the language of mathematics. I present the later Wittgenstein and the intuitionists as the most eminent challengers of the dogma showing that both have to offer valuable arguments against it. Inspired by Kolmorov’s interpretation of intuitionistic logic I examine the connection between intuitionistic logic and imperative logic. Along the way I offer a solution to Jørgensen’s Dilemma rejecting another dogma, the dogma based on the belief that there could not be a deduction in which premises and conclusion are something other than propositions.

KEYWORDS: mathematics, language, deduction, intuitionism, imperative logic, philosophy of mathematics, category.

1. Introduction

1.1. The dogma of the precedence of the descriptive uses of language

The view that asserting, and along with it, naming, are the most important uses of language has prevailed in logic, as well as in philosophy, for a long time. We shall name this view the dogma of the precedence of the descriptive uses of language. Our dogma should not be confused with the view according to which learning how to name objects and how to assert something about them comes before learning how to ask questions or state commands concerning these objects in the process of language acquisition. What is here meant by the term precedence is not precedence in the chronological sense, but precedence in the sense of being more basic, more fundamental. Wittgenstein was the first to draw attention to the fact that the answer to the question “Which uses of language are basic?” is relative to different needs and aims, theoretical as well as practical, that this language ought to serve. Following this idea of Wittgenstein we are not going to be concerned with language in general, but just with the language of mathematics. So the question is the following. Is giving precedence to the descriptive uses of the language of mathematics over others, such as commanding or deducing, warranted? In what follows we shall try to challenge the dogma and show that the answer to our question is negative. We shall present Wittgenstein’s philosophy of mathematics and Kolmogorov’s interpretation of intuitionistic logic in that context, arguing that both the later Wittgenstein and the intuitionists can be seen as our allies, that is, as the opponents of the dogma.

1.2. Roots of the dogma

The opinion that naming and asserting are the most important uses of the mathematical language has not always been the dominant view. Until the second half of the 19th century, the aim of mathematics was understood as having to do with calculation and problem solving. A great turn in a way of thinking
about mathematics was caused by what Keith Devlin called the *Göttingen Revolution.* (Devlin 2003) The revolution took place in Göttingen in the 1950s. One of the turning-points of the revolution was the transition to the extensional understanding of mathematical notions. According to the extensional understanding of meaning, words are like labels – they denote objects, and their meaning consists entirely in the objects they denote. Among these notions one is especially important and well known – the notion of function. Before the revolution a function was something closely connected to formulas of a certain kind, namely, equations, for instance, \( y = 2 + 3x - 5 \), and it was conceived as some sort of rule or algorithm that was supposed to take us from numbers signified by a variable ‘x’, called ‘arguments’, to numbers signified by ‘y’, called ‘values’. As a result of extensionalizing the notion of function, a function was reduced to a set of ordered pairs, the first projections of which are arguments, and the second projections are values. The rule itself became less important and all that mattered was the outcome of applying the rule – which values are assigned to which arguments. The concept of function thus became an abstract mathematical concept whose properties (such is, for instance, the property of always giving different values for different arguments, that is, the property of being an injection function) were the objects of mathematical investigation.

According to the post-revolutionary conception of mathematics, mathematics is an investigation of conceptual reality – reality of abstract mathematical objects (such as sets, functions, etc). The main aim of mathematical study is discovering truths about this conceptual world and mathematical questions are primarily concerned with the truth values of certain propositions concerning abstract objects in this world. Along with reshaping the way we see mathematics the revolution changed our views on the language of mathematics as well. Since the mathematical endeavor is concentrated around truth, mathematical language ought to be concentrated around propositions and thus asserting emerges as the most important use of this language. Along with asserting comes naming. In order to make certain assertions about these abstract objects, we must firstly denote them somehow.

The post-revolutionary picture of mathematics and its language is in spirit platonic. However, many philosophers, as well as working mathematicians, who were not fond of Platonism, have accepted this picture. Their rejection of the objective existence of mathematical objects was not in accordance with the precedence that was given to asserting, as a descriptive use of language. However, there have been some exceptions, to which we now turn to.

2. Calling the dogma into question

2.1. Wittgenstein’s philosophy of mathematics: mathematics is all about calculation

The first philosopher to criticise the descriptivistic picture of the language of mathematics after the revolution was the later Wittgenstein. His conception of mathematical language is closely connected with his view on meaning put forward in the *Philosophical Investigations.* Wittgenstein’s famous slogan “Don’t ask for the meaning, ask for the use” says that the meaning of words should be understood in relation to the activity (or in Wittgenstein’s words, *form of life*) in the context of which these words are used.\(^1\) Simply put, the way the word is used is what constitutes its meaning, not the objects that it denotes, contrary to the extensional view. Differing from the later Wittgenstein, Wittgenstein in the *Trac-

\(^1\) For more about Wittgenstein’s understanding of activity playing the prominent role in his understanding of meaning see: Luntley 2010, 30
tatus understands meaning extensionally, giving primacy to naming, and especially to asserting, over other language uses. In the Investigations Wittgenstein criticises the conception of language from the Tractatus according to which the primary function of language is to describe the world. According to the later Wittgenstein, it would be wrong to talk about the primary use of language in general, in some absolute sense. Which language use has primacy over others is relative to different activities, or in other words, forms of life. Along these lines, the answer to the question “Which language use has primacy in mathematics?” should be sought, Wittgenstein thinks, in the very practice of doing mathematics. Through the activity of doing mathematics mathematical symbols and sentences obtain their meaning. According to Wittgenstein, this activity is not connected with asserting, but with calculating. (Marion 1998, 4) In Friedrich Waismann’s notes we find the following words of Wittgenstein: “Mathematics is always a machine, a calculus. The calculus does not describe anything.” (Waismann 1979, 106) A little bit further on in his conversation with Waismann Wittgenstein, explaining what he means by a calculus, continues: “A calculus is an abacus, a calculator, a calculating machine.” (Waismann 1979, 106)

I think that the parallel Wittgenstein draws between the activity of practicing mathematics and using an abacus should not be taken literally. A formal system is also called a calculus, although strictly speaking, it does not have to calculate anything. By calling a formal system a calculus one does not wish to imply that this system has to do with numbers but rather that it is a sort of machine – a machine that given certain inputs, formulas, with the help of certain rules, inference rules, produces a result, a proof of these formulas. That is, a machine whose goal is producing theorems, that is, proving formulas (in a way similar to a calculator producing results by performing arithmetical operations on numbers). Wittgenstein’s stressing that mathematics is a calculus and that the primary activity of a mathematician is calculating could be interpreted as putting proving in the center of mathematical practice, as its most important activity, and accordingly, in the center of mathematical language. Such an interpretation is favoured by Wittgenstein’s claim that “The proof is a form of mathematical proposition.” (Wittgenstein 2001, 6.1264) This interpretation is in accordance with Wittgenstein’s slogan that the meaning is in the use. According to the slogan, the meaning of mathematical sentences is constituted by the role they play in mathematical activity. (Mathematical activity is a sort of framework - form of life or a linguistic game, where these linguistic expressions are used.) This activity, according to Wittgenstein, is not describing the platonic universe of mathematical objects, but proving certain formulas.

Wittgenstein was not the only one to relate the meaning of mathematical propositions with proving. That was also done by the intuitionists, to whom the next chapter will be devoted. Their attack on the dogma will be more severe than Wittgenstein’s because it is directed to the very edifice of classical logic.²

2.2. Intuitionism

Some philosophically oriented mathematicians tried, along with Wittgenstein, to understand mathematical language according to what they thought to be the nature of the object of mathematics. They believed that the world of mathematical objects does not exist independently of the human mind, but is rather a product of our mental construction. This philosophical standpoint is called constructivism. Constructivist views forced these mathematicians to conclude that, since mathematical objects do not have an objective existence, the correctness of mathematical propositions concerning these objects

² For a discussion of philosophical reasons standing behind choosing intuitionistic over classical logic in terms of the meaning of mathematical expressions and in the context of Wittgenstein’s slogan see Dummett 1973, 216
should not be seen as platonic truth, but as provability. These conclusions clashed with classical logic leading consequently to a discovery of a new one – intuitionistic logic. The most well-known point of intuitionists’ departure from classical logic is the rejecting the law of the excluded middle. According to the standard interpretation, intuitionists take provability as the assertoric force of a proposition, instead of truth. The formula A is then read as: “It is provable that A”, while the formula ¬A is read as: “It is provable that A implies ⊥” (the negation of A being defined as A implies ⊥). Such a reading of formulas gives us that law of the excluded middle, that is, the formula A∨¬A says: „It is provable that A is provable or it is provable that A implies ⊥“. Having that in mind, it becomes clear why the intuitionists reject law of the excluded middle – some formulas cannot be proven, nor can it be proved that they imply absurdity. Some other classical theorems that are also being rejected in intuitionistic logic are double negation law, that is, the formula ¬¬A → A (the converse is accepted, though) and Pierce’s law: ((A→B)→A)→A. Rejecting law of the excluded middle, the double negation law and Pierce’s law could be understood as basic, as the Hilbert-style axiomatic system for intuitionistic logic suggests since it differs from the classical variant in that in it the formulas A∨¬A, ¬¬A → A and ((A→B)→A)→A are not on the list of axiom schemes. In the natural deduction formulation of intuitionistic logic, disallowing law of the excluded middle, Pierce’s law and the double negation law is the result of rejecting the strong reductio ad absurdum and Pierce’s rule.

In what follows an interpretation of intuitionistic logic that was formulated by Kolmogorov will be presented. This interpretation will be especially interesting to us because it reexamines the dogma on other fronts also, independently of the relation to specially construtivistic reasons. _

3. Intuitionism and the uses of language

3.1. The calculus of problems

It has already been said that the intuitionists think mathematician should not be interested in truth, as distinct from provability. What a mathematician should be interested in is the possibility of constructing certain mathematical objects and proofs. In accordance with these views Kolmogorov proposed a reading of intuitionistic logic according to which the object of mathematics should not be propositions, but problems. The logic that fits mathematics thus understood – intuitionistic logic, should not be seen as a logic of propositions but as a logic of problems which Kolmogorov calls the calculus of problems.

Kolmogorov explains what he means by mathematical problems by giving the following examples:

1. Find four integers x, y, z, and n such that xn + yn = zn, where n>2
2. Construct* a circle passing through three given points (x, y, z).
(* "The permissible means of construction must be indicated when stating this problem", Kolmogorov remarks)
3. Prove that Fermat’s theorem is false. (Kolmogorov 1932, 151)

In Kolmogorov 1932 problems are denoted by letters a,b,c… Supposing that it is intuitively clear, on the basis of the given examples, what is meant by elementary mathematical problems, (Kolmogorov 1932, 152) Kolmogorov explains how out of less complex problems we can inductively build more complex ones by applying connectives. The inductive building is the same as in classical propositional
logic. If \( a \) and \( b \) are problems, then so is every formula built from these by applying the connectives conjunction (\( \land \)), implication (\( \rightarrow \)), negation (\( \neg \)) or disjunction (\( \lor \)). Formula \( a \land b \) denotes a problem saying: “Solve both problems \( a \) and \( b \)”, while \( a \lor b \) stands for the problem “Solve at least one of the problems \( a \) and \( b \)” Furthermore, the formula \( a \rightarrow b \) is the problem “Given a solution to problem \( a \), find a solution to problem \( b \)” or in other words: “Reduce the solution of problem \( b \) to the solution of problem \( a \)”. The problem denoted by \( \neg a \) says: “Assuming that there is a solution to problem \( a \), derive a contradiction”. (Kolmogorov 1932, 152)

It is not clear how the absurdity connective should be understood according to the Kolmogorov’s interpretation of the connectives presented above. It does not occur among the basic connectives of the calculus of problems. That in itself is not problematic because absurdity can be defined using negation and conjunction. However, Kolmogorov is facing a problem here because he chose to define negation in terms of absurdity and implication (remember that the problem \( \neg a \) stands for “Assuming that there is a solution to problem \( a \), derive a contradiction.”) and therefore he is not able to give a non-circular definition of absurdity. (Díez 2000a, 46) Kolmogorov’s way out would be to take absurdity as basic. For instance, the connective \( \bot \) could be interpreted as denoting the problem “Prove that 0=1”.

The axiomatization of the calculus of problems coincides with the axiomatization of Heyting-Brouwer intuitionistic logic, that is, with the Hilbert-style axiomatic system for intuitionistic logic. (Kolmogorov 1932, 155)

The axioms of the calculus of problems Kolmogorov sees as problems for which it is supposed that they have already been solved. Kolmogorov emphasizes that a problem can be supposed to be solved only if one has a general method for its solving applicable to cases of arbitrary problems \( a, b, c \ldots \) The formula \( a \lor \neg a \) (law of the excluded middle for problems) is not on the list of axioms because there is no general method on basis of which we could for every problem \( a \), either find the solution of \( a \) or draw a contradiction from the assumption that \( a \) has a solution. (Kolmogorov 1932, 156) The same holds for double negation law and Pierce’s law.

But what does it mean, in Kolmogorov’s terms, to solve a problem? Kolmogorov says “The aim of the calculus of problems is to develop a method allowing one to apply automatically a number of simple computational rules for solving [emphasis mine] a problem (...)”. (Kolmogorov 1932, 154) He further speaks about reducing the problem to the axioms by the help of these simple computation rules. So, “solving” a problem means reducing it to the problems that have already been solved. But what does it mean to reduce a problem and what are these computational rules? I think that computational rules are just inference rules of the calculus of problems and reducing a problem to the axioms is just deducing it from them. Therefore, by solving a mathematical problem \( a \) Kolmogorov means proving \( a \) (to be a theorem) in the calculus of problems.

Sometimes Kolmogorov speaks of solvability of problems as though it is the counterpart of truth in the calculus of problems. In order to show that the consequences of axioms are true, one first has to establish the truth of the axioms themselves. Kolmogorov claims that the same holds in the calculus of problems, in regard to solvability. To establish that the problems obtained from the axioms are solvable, one needs to show that the axioms themselves are solvable. (Kolmogorov 1932, 154) For these reasons it is hard to tell whether solvability (of mathematical problems) should be understood as a meta-notion that the theory (the calculus of problems) is supposed to explain or rather, a not fully developed semantic notion. These questions are not to be dealt with here. However, we shall return to the notion of solvability and solutions of problems later.
Which use of language should be dominant in Kolmogorov’s calculus of problems? Typical examples of problems Kolmogorov gives start with words: “Find…”, “Solve…”, “Prove that…” etc. So, problems are stated as commands of some sort. If mathematical problems Kolmogorov talks about can be understood as commands, it would mean that the calculus of problems (that is, thus interpreted intuitionistic logic) represents a logic of imperatives of some sort. (It should be noted that, while commanding is a use of language, imperative is a sentence that expresses a command. Since we do not say the logic of asserting, but the logic of propositions, we shall not be taking about the logic of commands but about the logic of imperatives.) In such logic one would expect commanding to take primacy over asserting. In what follows we shall see which language function the calculus of problems gives precedence to and in relation to that, whether intuitionistic logic is a good candidate for the logic of imperatives.

3.2. Problems seen as imperatives

3.2.1 The second dogma and Jørgensen’s dilemma

A consequence of the view according to which mathematical problems are commands would be that the calculus of problems gives us a somewhat different picture of deduction in mathematics from the one we are used to. Namely, imperatives would be premises and conclusions of deductions, not propositions. In connection with that, we shall consider what in the literature is considered to be the biggest obstacle for talking about deductions in which premises and conclusions are not propositions, but imperatives. (Hansen 2008, 77) That obstacle is better known as Jørgensen’s dilemma. It was first stated in Jørgensen 1938 in the following way: “So we have the following puzzle: According to a generally accepted definition of logical inference only sentences which are capable of being true or false can function as premises or conclusions in an inference; nevertheless it seems evident that a conclusion in the imperative mood may be drawn from the premises one of which, or both of which, are in the imperative mood. How is this puzzle to be dealt with?” (Hansen 2008, 5)

Instead of “logical inference” we shall use the term “deductive inference” or “deduction” for short. On one horn of the dilemma we have strong intuitions that from the premises some of which are imperatives a conclusion can be inferred that is itself an imperative, and moreover, that we make this kind of deductive inferences every day. For instance:

\[
P) \text{Buy (some) apples!}
P) \text{Buy (some) pears!}
\text{Therefore,}
\text{C} \text{ Buy (some) apples and pears!}
\]

\[
P) \text{If you run to a ‘Stop’ sign when driving, stop your car!}
P) \text{There is a ‘Stop’ sign in front of you.}
\text{Therefore,}
\text{C} \text{ Stop your car!}
\]

On the other horn of Jørgensen’s dilemma there is a “well accepted definition” of deduction as truth preservation. According to this definition, a deduction is a transition from premises to the conclusion such that
if the premises are true so is the conclusion, and therefore, we cannot have deductive inferences whose premises and conclusions are imperatives, because imperatives cannot be true (or false, for that matter). Jørgensen’s dilemma is, therefore, a dilemma about whether there could be a deduction from imperatives to imperatives. Some authors consider this dilemma a fundamental problem for the logic of imperatives. However, we are about to show that Jørgensen’s dilemma is not such a severe problem as it might appear at first glance.

Every characterization of deduction as some sort of truth preservation will be called an alethic characterization of deduction. The difficulty which the dilemma put us in comes from the fact that we cannot at the same time accept the alethic characterisation of deduction and the fact that we have deduction from imperatives to imperatives. This presents a difficulty as far as we believe, as most authors do, that the alethic conception of deduction is the only one.

This belief, however, has no real basis but presents a dogma. This is the second of the two dogmas that are criticised in this paper. What is the relation between them? The two dogmas go well together. If we follow the first one according to which naming and asserting are the most important functions of language, we shall be inclined to characterise other functions in terms of these. This holds for deduction as well. The most natural thing in that case (the case of characterising deduction in terms of the descriptive uses of language) would be to characterise deduction as some sort of transition from a proposition to a proposition (or, from a set of propositions to a set of propositions) that preserves truth. So, if we adhere to the first dogma, we shall probably accept the second one as well.

By rejecting the second dogma, Jørgensen’s dilemma disappears. This can be done in the following way. Instead of the alethic characterisation of deduction, we shall adhere to the categorial one, following Došen 2016. The categorial characterisation does not presuppose that premises and conclusions must be things that are truth-apt and is in accordance with having deductions from imperatives to imperatives.

3.2.2 Categorial characterization of deduction

A small category is a mathematical structure made out of two sets, such that the elements of the first one are called objects and the elements of the second one are called arrows, and of two functions from arrows to objects. One of them assigns to every arrow an object that is called its source and the other assigns to every arrow an object called its target. (We deal here only with small categories, and from now on by category we shall understand small category.) The arrow \( f \) having \( A \) as its source and \( B \) as its target is written \( f: A \rightarrow B \). In order for the described structure to make up a category, in addition to the two sets and functions already mentioned, the structure must contain certain operations on arrows and the equalities that hold for them. One of the two operations is the binary operation of composition which from the two arrows \( f: A \rightarrow B \) и \( g: B \rightarrow C \) makes the arrow \( g \circ f: A \rightarrow C \). The other is the nullary operation of identity, called the identity arrow, written \( 1_A: A \rightarrow A \).

The idea of the categorial characterization of deduction is to represent deductions as arrows in a category. Like an arrow, every deduction has its source – a premise (or a set of premises) and its target – a conclusion. To the identity operation (in a category) corresponds the trivial deduction from \( A \) to \( A \), where the premise and the conclusion are the same. To composition corresponds a simple form of cut in a sequent calculus. (Došen 2016, 70)

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3 If deduction from imperatives to imperatives is not possible, neither is the logic of imperatives that is supposed to describe this deduction.
Associativity is assumed to hold for composition; i.e., we have \( h \circ (g \circ f) = (h \circ g) \circ f \). Speaking in terms of deductions this equality, namely, \( h \circ (g \circ f) = (h \circ g) \circ f \), can be understood as an equality of deductions that concerns permuting cut with cut in sequent systems. (Došen 2016, 70) Moreover, in a category, one assumes identity laws, that is, laws of composing with identity arrows: \( f \circ 1_A = 1_B \circ f = f \). These laws could be understood as equalities between deductions – when we compose a deduction \( f \) with the trivial deduction either on the side of premise or on the side of conclusion, we get as a result the same deduction from which we started – the deduction \( f \). In a category, in addition to those above, one can have some further operations on arrows that correspond to connectives in logic. Such categories are called categories with additional structure. (Došen 2016, 71)

The above presented conception of deductions as arrows in a category should lead to a conclusion that, contrary to the second dogma, what determines deduction is not necessarily tied to the specificity of objects that make the premises and the conclusions (their truth-aptness, for instance). Deduction is determined by certain laws which it obeys. A deductive system makes up a structure – a category, and being that kind of structure is what is essential for being a deductive system.

Since the categorial characterisation of deduction does not forbid imperatives functioning as premises and conclusions of deductive inferences (or in other words, as objects in a category), we no more have an obstacle for speaking about deductive systems for imperatives, that is about the logic of imperatives.

### 3.3. Intuitionistic logic as a logic for imperatives

In this chapter we will be dealing with the question of whether the calculus of problems, that is, intuitionistic logic, gives adequate means for describing the rules of deduction between imperatives.

We shall consider the natural deduction formulation of intuitionistic logic, not the axiomatic one. Besides simplicity, the reason is that the goal of this paper is to challenge the dogmas and our choice of the formal system should reflect our attitude towards the dogmas. Axiomatic systems cohere with the dogmas, especially the first one, because they give precedence to the descriptive uses of language over the prescriptive ones. In axiomatic deductive systems axioms are preferred over rules. (We have many axioms and only one rule of inference in Hilbert-style axiomatic systems for classical and propositional logic.) Axioms are propositions. They have truth values, they assert something. Therefore they belong to the descriptive part of language. Inference rules, however, unlike axioms, do not assert anything. Rules command, give permissions, allow some conclusions to be made, but do not describe, as assertions do. They prescribe. In that sense, rules belong to the prescriptive part of language.

Having said that, we can now go back to the question that interests us here, namely, are natural deduction rules of the calculus of problems in accordance with the way connectives behave in deductions where imperatives are in the premises and conclusions?

It has been said that imperatives are sentences that express commands. In Vranas 2013 imperative formulas express not only commands but prescriptions in general (however, that does not make much difference for our present purposes) and they are obtained by applying the following formation rules:

- **R1:** If \( p \) is a formula of propositional logic, \( !p \) is an imperative formula
- **R2:** If \( i \) and \( j \) are imperative formulas, then so are \( \neg i, i \lor j, i \land j \).
- **R3:** If \( p \) is a propositional formula and \( i \) an imperative formula, then \( p \rightarrow i, i \rightarrow p, p \leftrightarrow i, i \leftrightarrow p \) are imperative formulas. (Vranas 2013, 2)
So, atomic imperative formulas are obtained by applying the imperative operator ! on any propositional formula A and they are of the form !A. The formula !A is read as “Let it be the case that A”.

The formula !A˄!B is read as “Let it be the case that A and let it be the case that B”. The imperative disjunction where the disjuncts are atomic formulas, that is, !A˅!B, is read as “Let it be the case that A or let it be the case that B” whereas imperative negation, where the negated formula is an atomic imperative formula, that is, the formula ¬!A, is read as ”Let it not be the case that A”. Formulas with implication as the main connective (called ‘conditional imperatives’), where the antecedent is a proposition p and the consequent an imperative i, that is the formulas of the form p→i are read as “If p is the case, then i” (the imperative “If you run to a ‘Stop’ sign, stop your car!” mentioned above is of this form). Imperatives of the form i→p are read “i only if p” (for instance, “Marry him only if you love him!”).

In Vranas 2013 we have as a theorem of imperative logic the formula !(А˅¬А). Intuitively, this formula says: “Let A or not A be the case.” If we accept the law of excluded middle (in the case of propositions), we would not mind this formula being a theorem, since, if A or not A is always the case, then whatever we do it will be in accordance with what the imperative !(А˅¬А) prescribes (since an imperative !A commands making A be the case). What about the formula !A˅!¬А? This formula is the imperative variant of law of excluded middle and it is read “Let A be the case or let not A be the case.” According to Vranas this formula expresses the same prescription as !(А˅¬А). In other words, these two formulas are interderivable, because ! passes through disjunction. So, !A˅!¬А is also a theorem in Vranas’ imperative logic. But it is not clear why this should be so. It appears that in some cases there is neither a command to do something nor the command not to do it. For example, let A stand for a proposition: “A griffon vulture is making a nest on the roof of your house right now”. Even if we accept the interderivability between ¬A and !¬A, that is, accept that !¬A can be read as “Do not let A be the case”, that still does not make a case for the law of excluded middle. The imperative “Let the griffon vulture make a nest on the roof of your house or do not let such a thing happen” does not sound much better either. It is neither commanded nor forbidden to let the griffon vulture make his nest.

So it seems that in the logic of imperatives one should not have the variant of law of excluded middle. (In other words, a natural deduction deductive system for imperatives should contain neither the strong reductio ad absurdum nor the Pierce’s rule.) This presents a strong reason in favour of choosing intuitionistic inference rules for imperative connectives instead of classical ones. However, the introduction rule for disjunction presents a strong reason for doubt. This rule says that from the premise A one can infer А˅B. So, if this rule is to be accepted in imperative logic, one will have as a consequence !A ├ !А˅!B, that is, i ├ i˅j, in a more general case (as we, in fact, do have in Vranas’ imperative logic; moreover, since ! passes through disjunction, in Vranas’ logic we also have !A ├ !(А˅B)). However, this is in sharp contrast to our intuitions concerning which prescription follows from another, as the following example should show. Let A be the proposition “The letter has been sent” and B the proposition “The letter has been burned”. In such a case we would not want to say that !A˅!B follows from !A. “Send the letter” does not imply “Send the letter or burn it”. (The example is from Ross 1944, 38.) The first of the imperatives being commanded does not suffice for a second to be as well.

But, why is this so? What is the source of the force of Ross’ counterexample? In deontic logic, unlike in Vranas’ imperative logic, commands are distinguished from prescriptions in general. Prescriptions are of three kinds. Namely, something can be forbidden, permitted, or commanded, that is, in other words, obligatory. These three kinds of prescriptions are represented by three kinds of operators and they are

4 In other words, it is trivially satisfied. We shall speak about imperative satisfaction later.
related to each other in certain ways. In deontic logic if one is obliged to make A be the case or to make B be the case (for instance, to make a right turn or to make a left turn), then she is permitted to do any of the two alternatives (but not necessarily both). Imperatives express commands, that is, obligations. According to deontic logic (which I think nicely explains our pre-theoretical reasoning about prescriptions concerning this point), if one is commanded to send the letter or to burn it, she may burn the letter (as well as sending it). Bearing that in mind, the problem with disjunction introduction becomes the following. If we accept the disjunction introduction rule we must accept that from the prescription “Send the letter” it follows that one may burn it, and moreover, that one may do whatever with it. In general, the consequence of disjunction introduction would be that if something is commanded, anything is permitted. And this is not something we would want in our imperative logic.

How does all of this relate to our question? The discussion of the Ross’ example is supposed to lead us to the conclusion that intuitionistic logic, that is, the calculus of problems, should not be interpreted as describing deductions whose premises and conclusions are mathematical problems seen as imperatives. In general, every deductive system in which we have $A \vdash A \lor B$ can not be an adequate representation of deductions whose premises and conclusions are imperatives.

One can here object that invalidating disjunction introduction for imperatives in general does not show that mathematical problems could not be seen as imperatives of a special kind such that $!A \vdash !A \lor !B$ holds. But this does not seem to be a viable option, since “Find the square root of 9” does not yield “Find the square root of 9 or prove Fermat’s theorem” any more than “Send the letter” implies “Send the letter or burn it”.

Ross emphasized that there are, however, two ways in which deduction of imperatives could be understood and in accordance with it, that there are two logics for imperatives. The first conception of deduction of imperatives is related to the question whether an imperative is satisfied and the other with the question whether an imperative is in force. (Segerberg 1990, 203) When talking about satisfaction of an imperative we are in the field of semantics. As propositions have their truth values, imperatives have their satisfaction values. However, in Vranas’ opinion, in imperative logic one should not have two, but three satisfaction values: an imperative can be satisfied, violated or avoided. (Vranas 2008, 4) The latter being an option only in the case of conditional imperatives. In Ross 1944 an imperative is satisfied if the corresponding proposition to which the prescription is related is true. (Ross 1944, 36) This way of understanding satisfaction conditions is in accordance with Vranas’. For Vranas, as well as for most authors working in the field of imperative logic, an imperative $!A$ is satisfied if the proposition $A$ is true. Furthermore, $i \land j$ is satisfied when both $i$ and $j$ are satisfied; $i \lor j$ is satisfied when at least one of them is, and $\neg i$ when $i$ is violated. A conditional imperative whose antecedent is a proposition is said to be violated if the antecedent is true and the consequent violated, satisfied if the antecedent is true and the consequent satisfied and avoided when the antecedent is false or the consequent is avoided. (Vranas 2008, 18-19)

In Vranas’ logic, satisfaction is connected with the notion of semantic consequence. The imperative $j$ is a semantic consequence of the set of imperatives $i_1, \ldots, i_n$ if in every model where all the imperatives $i_1, \ldots, i_n$ are satisfied, the imperative $j$ is also satisfied. (Vranas 2013, 6) If one reduces the deduction of imperatives to the semantic consequence relation defined in this way, that is, if one stipulates that there is a deduction $f$: $f: i_1, \ldots, i_n \vdash j$ if $j$ is a semantic consequence of $i_1, \ldots, i_n$, then the logic of imperatives that represents such a deduction, is something Ross called the logic of satisfaction. (Segerberg 1990, 203) According to such a logic, “[...] to infer one imperative from another means to say something about a

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5 There are exceptions, however. See Segerberg 1990.
necessary connection between the satisfaction of imperatives in question.” (Ross 1944, 37) This logic differs from the logic of validity. (Segerberg 1990, 203) While the former is concerned with what one can, from hypotheses about satisfaction of some imperatives, infer about the satisfaction of others, the latter is concerned with what prescription can be inferred from others. The fact that these two logics do not coincide becomes evident in the case of disjunction introduction for imperatives discussed above.

From the fact that the imperative \(i\) is satisfied it follows that the imperative \(i \lor j\) is satisfied also. We also have that if \(!A\) is satisfied, so is \(!!(A \lor B)\), because if \(!A\) is satisfied, A is true and therefore, so is \(A \lor B\). And if \(A \lor B\) is true, the imperative \(!!(A \lor B)\) is satisfied. Therefore, in the logic of satisfaction one has a deduction \(g: !A \vdash !!(A \lor B)\) and a deduction \(f: i \vdash i \lor j\). However, this should not be the case in the logic of validity. If “Send the letter!” has been commanded, that does not mean “Send the letter or burn it!” has been commanded too. The later prescription does not follow from the first for the reasons already mentioned. Therefore, a deductive system in which we have \(i \vdash i \lor j\) or \(!A \vdash !!(A \lor B)\) can be an adequate logic for imperatives only as a logic of satisfaction, which in fact, is not concerned with deduction between imperatives (as the logic of validity is), but only with the semantics for imperatives.

3.4. Proofs and solutions to problems

The calculus of problems, therefore, does not describe deduction from imperatives to imperatives, since in this calculus we have \(a \vdash a \lor b\). But, what can accepting \(a \vdash a \lor b\) tell us about the conception of deduction of problems standing behind the calculus of problems? For what reason does Kolmogorov think the problem \(a \lor b\) should follow from the problem \(a\)? Is it the case that Kolmogorov, like in the logic of satisfaction, implicitly identifies a deduction from a problem \(c\) to a problem \(d\) with some relation \(R\) that holds between \(c\) and \(d\), similar to the Vranas’ relation of semantic consequence, such that one has \(a \Re (a \lor b)\)?

As semantic consequence is related to the semantic notion of satisfaction, so one might expect the \(R\) relation to be related to the semantic notion of solvability of mathematical problems Kolmogorov talks about. We could say that one has \(a \Re c\) whenever it is the case that if \(a\) has a solution, \(c\) has a solution also. If the problem \(a \lor b\) has been solved if at least one the problems \(a\) and \(b\) has been solved, in the calculus of problems one would have \(a \Re (a \lor b)\), since when \(a\) has a solution, then at least one of the problems \(a\) or \(b\) has a solution, and accordingly, the problem \(a \lor b\) has a solution.

If problems are understood as imperatives, solvability of a problem could be reduced to satisfaction of a corresponding imperative stating that problem. Assuming atomic imperatives are of the form \(!A\), as in Vranas 2013, one could say that the elementary problem \(a\) has been solved iff the corresponding imperative (the one stating that problem) \(!A\) has been satisfied, that is, if the proposition \(A\) is true. For instance, the mathematical problem “Prove Fermat’s theorem” understood as an imperative in the style of Vranas actually says: “Let ‘Fermat’s theorem has been proved’ be the case.” The given mathematical problem, thus understood, would have a solution only if it were the case that Fermat’s theorem has been proved.

According to the view proposed above, solutions of mathematical problems would be expressed in the form of propositions, but propositions not referring to platonistic entities, but to the facts that certain human mental actions that result in proofs and construction have been carried out. Then, the above mentioned \(R\) relation would be the consequence relation between formulas of propositional logic. Deduction from a problem \(a\) expressed by an imperative \(i\) to problem \(b\) expressed by an imperative \(j\) would be accordingly defined in the following way. There is a deduction \(f: a \vdash b\) if in every model in which \(i\) is satisfied, \(j\) is also satisfied.
When he talks about solutions of problems Kolmogorov stresses that “the fact that I have solved a problem is a purely subjective one of no general interest in itself. Logical and mathematical problems, however, possess a special property of universal validity of their solutions, that is if I have solved a logical or a mathematical problem then I can present this solution in a commonly accepted way, and this solution must necessarily be recognized as being correct (…)”. (Kolmogorov 1932, 153) The first sentence of the cited passage suggests that Kolmogorov would disagree on reducing solvability of a problem to the satisfaction of an imperative, that is, to the truth of a proposition that states some subject has carried out a certain action (of writing some formulas on a paper). The very solution, as the proof itself, is not reducible to this action; it possesses objective and universal validity. Moreover, the cited passage shows that Kolmogorov considers the solution itself to be more important than the mere fact the problem in question has been solved and therefore, cannot be reduced to it. In a footnote at the end of the cited paragraph Kolmogorov adds that “the same is literally true for proofs of theoretical propositions”. (Kolmogorov 1932, 153, ft. 8) So, it appears that solutions are more akin to proofs (of propositions), rather than to propositions themselves. Namely, it is not clear what would it mean to present a proposition in a commonly accepted way. This is the way we usually talk about proofs.

But do we have enough evidence to equate solutions with proofs? When talking about the relation between propositional logic and the calculus of problems Kolmogorov says: “Along with the development of theoretical logic, which systematizes the schemes of proofs of theoretical truths, it is also possible to systematize the schemes of solutions of problems [emphasis mine], for example, geometrical construction problems. (…) By introducing the appropriate system of symbols, we can develop a formal calculus enabling us to construct symbolically system of such solution schemes. Thus a new calculus of problems arises along with theoretical logic.” (Kolmogorov 1932, 151) So it appears that solutions of problems are proofs of formulas in the calculus of problems. In the light of that, we could explain what solvability of a problem amounts to. Namely, one could say that a problem a has been solved if the proof of the formula a has been given in the framework of the calculus of problems.

When talking about proofs, one should distinguish between categorical proofs, proofs from an empty set of hypotheses, and hypothetical proofs, proofs from hypotheses, where the set of hypotheses need not be empty. In the footnote mentioned above, Kolmogorov speaks about a certain disanalogy between solutions of problems and proofs of formulas of propositional logic: “It is, however, essential that every proved proposition is true; for problems there is no such notion of truth.” (Kolmogorov 1932, 153-154) It is evident here that by solutions Kolmogorov means categorical, not hypothetical proofs. Unlike the formulas proven from an empty set of hypotheses, i.e., theorems of intuitionistic logic, the formulas that are proven from hypotheses need not be true.7 (For instance, p˄¬p can be proven hypothetically, from hypothesis (p˄¬p)˄q in the natural deduction formulation of intuitionistic logic.)

Although what Kolmogorov is interested in are categorical, not hypothetical proofs, it should be noted that categorical proofs are just special cases of hypothetical ones. Namely, we can take the notion of hypothetical proof of B from A as basic, that is, f: A ⊢ B, and then define the categorical proof of B as its special case – the case where the set of hypotheses is empty, that is, f: ⊤ ⊢ B. (Došen 2015, 151) A hypothetical proof of B from A simply amounts to a deduction from A to B.

6 This objective component commands and questions are supposed to have has usually been overlooked in the literature. It is generally assumed that what is essential of commands and questions is that there is someone who states the command or poses the question and someone to whom that command or question is stated or posed. For instance, see Harrah 2002 and Belnap 1976.

7 Since we have validity, all theorems of intuitionistic logic are tautologies.
3.5. **Deducing as the most important use of the language of mathematics**

What does the above presented discussion tell us about deduction from a problem to a problem? It appears that for Kolmogorov there is a deduction from a problem \( a \) to a problem \( b \) only if there is a deduction from a solution of the problem \( a \) to a solution of the problem \( b \). In that sense, premises and conclusions of deduction that the calculus of problems describes are not really problems but rather their solutions, that is, categorical proofs. Taking the notion of hypothetical proof as basic and categorical proof as defined, reveals that the categorical proofs are special cases of hypothetical ones, in other words, that they are special cases of *deductions*. So, solutions of problems are in fact deductions.

But how can deductions function as premises and conclusions of other deductions? So far we have seen that the places of premises and of conclusions are not reserved for propositions exclusively. What about deductions? Let us take the natural deduction rule for implication introduction:

\[
\begin{array}{c}
A \\
\vdots \\
\vdots \\
\hline
B
\end{array}
\Rightarrow A \rightarrow B
\]

This rule says that when from hypothesis \( A \) we deduce \( B \), we can infer \( A \rightarrow B \) and cross out the hypothesis \( A \). What is the premise of this rule? It appears that it is not \( A \) itself, nor \( B \), but rather the *deduction* from \( A \) to \( B \).

In Došen 1985 it has been shown how one can, in the framework of sequent systems, formulate deductions whose premises and conclusions are also deductions. Such deductions are represented by sequents of higher levels. Higher level sequents, that is, sequents of level \( n+2 \), have sequents of level \( n+1 \) on the left-hand side and on the right-hand side of the turnstile. Sequents of level 1 are just ordinary sequents. Sequent \( \Gamma \vdash \Delta \) of level \( n+1 \) can be understood as a deduction whose premises are formulas in \( \Gamma \) and whose conclusion is a formula in \( \Delta \). (Došen 1985, 166) When \( \Gamma \) is empty, \( \Delta \) is a theorem, and when \( \Delta \) is empty, the deduction in question can be understood as a refutation of one of the premises in \( \Gamma \). In this way sequents of higher levels can be understood as deductions whose premises and conclusions are deductions as well.

So far we have presented an interpretation of Kolmogorov according to which he reduced a deduction of problems to deduction of their solutions. We have tried to show that *solutions* of problems can be understood as deductions. Therefore, the deduction Kolmogorov’s calculus of problems is concerned with is in fact a deduction of deductions. According to this picture of the calculus of problems, primacy is given to deducing over commanding. Moreover, in intuitionistic logic presented in this way, deducing is seen as *the* primary use of language in mathematics.

3.6. **Deduction and rules**

The categorial characterization of deduction has shown that deduction is characterized by certain *laws* which it obeys. At the level of deductive systems, the bearers of these laws are the inference *rules*. In that sense, when speaking about deductions, one should not overlook the importance of rules. Rules are *prescriptions*, they prescribe something – they tell us what we may and may not do, they give per-
missions and commands, they forbid. For instance, the natural deduction inference rule for conjunction elimination gives one the permission to conclude A from the premise A\( \land \)B. When expressing a rule one uses language in a prescriptive, not in the descriptive manner. That does not mean, however, that rules and prescriptions in general cannot be expressed by indicative sentences as when we say: “You must not forget that.” to state a prohibition. (Došen 2013, 18) Also, that does not mean prescriptions cannot be described, as when the command ”Fasten your seatbelt” is described by the words “The law prescribes to fasten your seatbelt.” The point we wish to stress here is rather that the prescriptive component of a rule is what is crucial for it to be applied and one cannot just describe the rule without losing this prescriptive component. Therefore, rules cannot be reduced to their descriptions. This has been noticed long ago by Lewis Carroll in his famous story about Achilles and the Tortoise. 

That rules and, therefore, prescriptive uses of language are more important for deduction, that is, for the language in which we speak about deduction, than the descriptive ones is manifested by the fact that one can have deductive systems without any axioms, like natural deduction, but not systems without at least one rule of inference. The reason standing behind this is that in logic inference rules are more basic than axioms. Namely, an axiom can be seen as a limiting case of inference rules, a case where we infer a proposition from an empty set of hypotheses. Therefore, one can say that for deduction prescribing is more important than asserting.

In everyday speech, rules are most often expressed by imperative sentences. However, in logic, this is not the case. Rules are formulated in the meta-language, as when the elimination rule for conjunction is expressed by the words: From the premise A\( \land \)B one can infer A. The same rule can also be symbolically represented by a line which signifies a permission to pass from the premise(s) to the conclusion:

\[
\begin{array}{c}
A \land B \\
\hline
A
\end{array}
\]

Bearing in mind the connection between deduction and rules, it could be said that deducing is connected with the prescriptive uses of language. In other words, in a language where deduction is the primary language use, prescriptive uses will also play an important role. This is opposed to what the two dogmas, which we have been considering, are telling us. According to the second dogma, giving account of deduction is inseparably connected with asserting, not with prescribing. According to the first, in the language of mathematics, and in logic in general, asserting has primacy over prescriptive functions. By giving precedence to deduction, and thus also to prescriptive functions of language, the calculus of problems presents intuitionistic logic as being opposed to both dogmas.

4. Conclusion

In this paper, we have tried to call into question the deeply rooted assumption that the most important uses of the language of mathematics and logic in general, are asserting and naming. We have seen

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\(^8\) Some authors that Parsons calls “cognitivists” would say sentences “Fasten your seatbelt” and “The law prescribes you fasten your seatbelt” differ only grammatically. Cognitivists think imperatives have the same meaning as the corresponding declarative sentences. Therefore, they can be equated with these sentences and consequently, have truth values. (Parsons 2012, 49) In Hofstadter 1939 the authors argue for the similar theses, supporting it with some formal results. One can object to these authors that they make the same mistake as the ones who think rules of inference can be expresses by axioms.
how, in the light of intuitionistic logic, the language of mathematics can be seen as based on deducing and prescriptive language uses. Wittgenstein’s philosophy of mathematics has also been presented in this light. We have seen that for Wittgenstein proving can be viewed as the form of life mathematicians live in. Bearing in mind the connection between deduction and proof, we have presented in this paper, we can now say that for Wittgenstein that form of life is in fact deducing.

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**Upotreba matematičkog jezika**

(Apstrakt)

U ovom radu kritikuje se dogma da su tvrđenje i imenovanje najvažnije upotrebe matematičkog jezika. Kasni Vitgenštajn i intuicionisti se prikazuju kao najistaknutiji kritičari dogme koji iznose jake argumente protiv nje. Istražuje se veza između intuicionističke logike i logike imperativa, pri čemu je istraživanje inspirisano Kolmogorovljevom interpretacijom intuicionističke logike. Na tom putu biće ponuđeno rešenje Jergensonove dileme na temelju odbacivanje jedne druge dogme, koja je u vezi sa već pomenutom, a koja se zaniva na uverenju da premise i zaključci dedukcija mogu biti isključivo iskazi.

Ključne reči: matematika, jezik, dedukcija, intuicionizam, logika imperativa, filozofija matematike, kategorija