Lebesgue Points of Multi-Dimensional Functions

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Abstract: Lebesgue and Walsh-Lebesgue points are introduced for higher dimensional functions and it is proved that a.e. point is a (Walsh)-Lebesgue point of a function \( f \) from the space \( L(\log L)^{d-1} \). Every function \( f \in L(\log L)^{d-1} \) is Fejér summable at each (Walsh)-Lebesgue point.

Keywords: Lebesgue point, Walsh-Lebesgue point, Walsh functions, Fejér-summability.

1 Introduction

It was proved by Fejér [1] that the \((C,1)\) or Fejér means of the one-dimensional trigonometric Fourier series of a continuous function converge uniformly to the function. The same problem for integrable functions was investigated by Lebesgue [2]. He proved that for every integrable function \( f \),

\[
\frac{1}{n+1} \sum_{k=0}^{n} s_k f(x) \to f(x) \quad \text{as} \quad n \to \infty
\]

at each Lebesgue point of \( f \), where \( s_k f \) denotes the \( k \)th partial sum of the Fourier series of \( f \). Almost every point is a Lebesgue point of \( f \) (see Zygmund [3] or Butzer and Nessel [4]).

The concept of Lebesgue points was extended to the one-dimensional Walsh system by the author in [5], the points are called Walsh-Lebesgue points in this case. The definition of Walsh-Lebesgue points is not a simple adaptation of the one of Lebesgue points, it needs new ideas, because the Walsh-Fejér kernels differ entirely from the trigonometric Fejér kernel. It was proved there that a.e. point
is a Walsh-Lebesgue point of a one-dimensional integrable function. Moreover, the Fejér means of the Walsh-Fourier series of \( f \in L_1[0,1] \) converge to \( f \) at each Walsh-Lebesgue point. The a.e. convergence of the Fejér means was proved earlier by Fine [6] (see also Schipp [7]).

In this paper we generalize the definition of Lebesgue and Walsh-Lebesgue points for higher dimensions. We prove that a.e. point is a (Walsh)-Lebesgue point of \( f \in L^{(\log L)^{d-1}} \). The Fejér means of the Walsh-Fourier series of \( f \in L^{(\log L)^{d-1}} \) converge to \( f \) at each (Walsh)-Lebesgue point.

2 Lebesgue Points

For a set \( \mathbb{X} \neq \emptyset \) let \( \mathbb{X}^d \) be its Cartesian product \( \mathbb{X} \times \ldots \times \mathbb{X} \) taken with itself \( d \)-times. We briefly write \( L_p(\mathbb{X}^d) \) instead of \( L_p(\mathbb{X}^d, \lambda) \) space equipped with the norm (or quasi-norm) \( \|f\|_p := (\int_{\mathbb{X}^d} |f|^p d\lambda)^{1/p} \) \((0 < p \leq \infty)\), where \( \lambda \) is the Lebesgue measure and \( \mathbb{X} \) denotes the torus \( \mathbb{T} = [-1/2, 1/2] \) or the unit interval \([0,1])\).

In the one-dimensional case Lebesgue differentiation theorem says that

\[
\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) \, dt = f(x)
\]

for a.e. \( x \in \mathbb{T} \), where \( f \in L_1(\mathbb{T}) \). This motivates the next definition. A point \( x \in \mathbb{T} \) is called a Lebesgue point of a function \( f \) if

\[
\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| \, dt = 0.
\]

Using Lebesgue differentiation theorem we can prove in the usual way that a.e. point \( x \in \mathbb{T} \) is a Lebesgue point of \( f \in L_1(\mathbb{T}) \) (see e.g. Butzer and Nessel [4] or Stein and Weiss [8]).

Feichtinger and Weisz [9] extended the definition of Lebesgue points to higher dimensions as follows. The strong Hardy-Littlewood maximal function is defined by

\[
M_x f(x) := \sup_{I \subset \mathbb{T}^d} \frac{1}{|I|} \int_I |f| \, d\lambda,
\]

where \( f \in L_1(\mathbb{T}^d) \), \( x \in \mathbb{T}^d \) and the supremum is taken over all rectangles \( I \subset \mathbb{T}^d \) with sides parallel to the axes. It is known that in the one-dimensional case the maximal function is of weak type \((1,1)\), i.e.,

\[
\sup_{\rho > 0} \rho \lambda \{ M_x f > \rho \} \leq C_1 \|f\|_1, \quad (f \in L_1(\mathbb{T})).
\]
However, for higher dimensions there is a function \( f \in L_1(\mathbb{T}^d) \) such that \( M_s f = \infty \) a.e. Thus \( M_s \) cannot be of weak type \((1, 1)\) if \( d > 1 \), but we have

\[
\sup_{\rho > 0} \rho \lambda(M_s f > \rho) \leq C_d + C_d \|f\|_{L_1(\log L)^{d-1}},
\]

where \( C_d \) is depending only on \( d \). Moreover,

\[
\|M_s f\|_p \leq C_p \|f\|_p \quad (f \in L_p(\mathbb{T}^d), 1 < p \leq \infty, d \geq 1).
\]

For these results see Zygmund [3], Stein [10] or Weisz [11, p. 71]. Set \( \log^+ u = 1_{\{u > 1\}} \log u \). Recall that a function \( f \) is in the set \( L_1(\log L)^k(\mathbb{T}^d) \) if

\[
\|f\|_{L_1(\log L)^k} := \int_{\mathbb{T}^d} |f(\log^+ |f|)^k d\lambda < \infty.
\]

If \( k = 0 \) then \( L_1(\log L)^k(\mathbb{T}^d) = L_1(\mathbb{T}^d) \). We can say that the role of \( L_1(\mathbb{T}) \) in one dimension is played in higher dimensions by \( L_1(\log L)^{d-1}(\mathbb{T}^d) \).

Inequalities (1) and (2) imply

\[
\lim_{h \to 0} \frac{1}{\prod_{j=1}^d h_j} \int_{x_1}^{x_1 + h_1} \cdots \int_{x_d}^{x_d + h_d} f(t) dt = f(x)
\]

for a.e. \( x \in \mathbb{T}^d \), where \( f \in L_1(\log L)^{d-1}(\mathbb{T}^d) \) or \( f \in L_p(\mathbb{T}^d) \) \((1 < p \leq \infty)\). Note that \( L_1(\log L)^{d-1}(\mathbb{T}^d) \supseteq L_p(\mathbb{T}^d) \) \((1 < p \leq \infty)\). Here \( h \to 0 \) is understood in the Pringsheim’s sense, i.e., \( h_j \to 0 \) for all \( j = 1, \ldots, d \).

A point \( x \in \mathbb{T}^d \) is called a Lebesgue point of \( f \) if \( M_s f(x) \) is finite and

\[
\lim_{h \to 0} \frac{1}{\prod_{j=1}^d h_j} \int_{x_1}^{x_1 + h_1} \cdots \int_{x_d}^{x_d + h_d} |f(t) - f(x)| dt = 0.
\]

The next theorem is proved in Feichtinger and Weisz [9].

**Theorem 1** Almost every point \( x \in \mathbb{T}^d \) is a Lebesgue point of \( f \in L_1(\log L)^{d-1}(\mathbb{T}^d) \).

### 3 Fejér Means of Fourier Series

For a one-dimensional integrable function \( f \) the \( n \)th Fourier coefficient is defined by

\[
\hat{f}(n) = \int_{\mathbb{T}} f(t) e^{-2\pi i nt} dt \quad (n \in \mathbb{Z}).
\]
The $n$th partial sum of the trigonometric Fourier series of $f$ is given by

$$s_n f(x) := \sum_{k=-n}^{n} \hat{f}(k)e^{2\pi ikx} \quad (n \in \mathbb{N}).$$

One of the deepest results in harmonic analysis is Carleson’s theorem [12, 13]:

$$s_n f \rightarrow f \quad \text{a.e. as } n \rightarrow \infty,$$

whenever $f \in L_p(\mathbb{T})$ ($1 < p < \infty$). This theorem does not hold, if $p = 1$. However, some summability results can be obtained in this case, too.

The Fejér-means of $f$ are defined by

$$\sigma_n f(x) := \frac{1}{n+1} \sum_{k=0}^{n} s_k f(x) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \hat{f}(k)e^{2\pi ikx} = \int_{\mathbb{T}} f(t) K_n(x+t) dt,$$

where $x \in \mathbb{T}, n \in \mathbb{N}$ and $K_n$ denote the Fejér kernels. As mentioned in the introduction, Lebesgue [2] proved for all $f \in L_1(\mathbb{T})$ that

$$\sigma_n f \rightarrow f \quad \text{at each Lebesgue point of } f \text{ as } n \rightarrow \infty.$$

In the multi-dimensional case let the $n$th Fourier coefficient of a function $f \in L_1(\mathbb{T}^d)$ be defined by

$$\hat{f}(n) = \int_{\mathbb{T}^d} f(t)e^{-2\pi i u \cdot t} dt \quad (n \in \mathbb{Z}^d),$$

where $u \cdot x := \sum_{k=1}^{d} u_k x_k$, $(x = (x_1, \ldots, x_d) \in \mathbb{R}^d, u = (u_1, \ldots, u_d) \in \mathbb{R}^d)$. Denote by $s_n f$ the $n$th partial sum of the trigonometric Fourier series of $f$:

$$s_n f(x) := \sum_{j=1}^{d} \sum_{k_j = -n_j}^{n_j} \hat{f}(k) e^{2\pi ik \cdot x} \quad (n \in \mathbb{N}^d).$$

Under $\sum_{j=1}^{d} \sum_{k_j = -n_j}^{n_j}$ we mean the sum $\sum_{k_1 = -n_1}^{n_1} \cdots \sum_{k_d = -n_d}^{n_d}$.

Carleson’s result does not hold for higher dimensions (see Fefferman [14]). The only known result is that

$$s_{n_1, \ldots, n_d} f \rightarrow f \quad \text{a.e. as } n \rightarrow \infty,$$

whenever $f \in L_p(\mathbb{T}^d)$ ($1 < p < \infty$) (Fefferman [15]).
Now we introduce the *Fejér-means* of \( f \) by
\[
\sigma_n f(x) := \frac{1}{\prod_{i=1}^{d}(n_i + 1)} \sum_{j=1}^{d} \sum_{k=0}^{n_j} s_k f(x) = \sum_{j=1}^{d} \sum_{k=-n_j}^{n_j} \left( \prod_{i=1}^{d} \left( 1 - \frac{|k_i|}{n_i + 1} \right) \right) \hat{f}(k) e^{2\pi ik \cdot x},
\]
\((x \in \mathbb{T}^d, n \in \mathbb{N}^d)\). In the following theorem we generalize Lebesgue’s theorem just mentioned (see Feichtinger and Weisz [9]).

**Theorem 2** For all Lebesgue points of \( f \in L_1(\log L)^{d-1}(\mathbb{T}^d) \) we have
\[
\lim_{n \to \infty} \sigma_n f(x) = f(x).
\]

### 4 Walsh-Lebesgue Points

The definition of the Walsh-Lebesgue points should fulfill the next two requirements: a.e. point is a Walsh-Lebesgue point of an integrable function and the Walsh-Fejér means of an integrable function converge at all Walsh-Lebesgue points. The proof of the one-dimensional version of Theorem 2 is based on the fact, that the Fejér kernels \( K_n \) can be estimated by an integrable, on \([0,1/2]\) non-increasing function \( K'_n \) such that \( \|K'_n\|_1 \leq C \) for all \( n \in \mathbb{N} \). Recall that
\[
K'_n(x) = C(n + 1) \mathbf{1}_{[0,1/(n+1)]} + \frac{C}{(n+1)x^2} \mathbf{1}_{[1/(n+1),1/2]}.
\]

This does not hold for the Walsh-Fejér kernels \( K_{2^n} \) (for the definition see the next section), because
\[
K_{2^n}(x) = \frac{1}{2} \left( 2^{-n} D_{2^n}(x) + \sum_{k=0}^{n} 2^{k-n} D_{2^n}(x+e_k) \right),
\]
where
\[
D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in [0,2^{-n}), \\ 0, & \text{if } x \in [2^{-n},1) \end{cases}
\]
are the Walsh-Dirichlet kernels, \( + \) denotes the dyadic addition and \( e_k := 2^{-k-1} \).

It is easy to see that if \( K_n' \) denotes the smallest non-increasing function for which \( K_n \leq K_n' \) then \( \|K_{2^n}'\|_1 = Cn \). Because of this difference of the Fejér and Walsh-Fejér kernels, a new definition of Lebesgue points is needed in the dyadic case.

By a *dyadic interval* we mean one of the form \([k2^{-n},(k+1)2^{-n})\) for some \( k,n \in \mathbb{N}, 0 \leq k < 2^n \). Given \( n \in \mathbb{N} \) and \( x \in [0,1) \) let \( I_n(x) \) be the dyadic interval of
length $2^{-n}$ which contains $x$. A Cartesian product of $d$ dyadic intervals is called a dyadic rectangle. For $n \in \mathbb{N}^d$ and $x \in [0, 1)^d$ let $I_n(x) := I_{n_1}(x_1) \times \ldots \times I_{n_d}(x_d)$, where $n = (n_1, \ldots, n_d)$ and $x = (x_1, \ldots, x_d)$. The $\sigma$-algebra generated by the dyadic rectangles $\{I_n(x) : x \in [0, 1)^d\}$ will be denoted by $\mathcal{F}_n (n \in \mathbb{N}^d)$. Let $E_n$ denote the conditional expectation operator with respect to $\mathcal{F}_n$. Obviously, if $f \in L^1[0, 1)^d$ then $(E_n f, n \in \mathbb{N}^d)$ is a martingale.

Butzer and Wagner [16] introduced the dyadic derivative of $f$ with the limit of

$$
\mathbf{d}_n f(x) := \sum_{k=0}^{n-1} 2^k (f(x) - f(x + e_k)) \quad (x \in [0, 1))
$$

as $n \to \infty$. For $f \in L^1[0, 1)$ let $F(x) := \int_{I_n(x)} f$ and investigate the function

$$
\mathbf{d}_n F(x) = \sum_{k=0}^{n-1} 2^k \left( \int_{I_n(x)} f - \int_{I_n(x + e_k)} f \right).
$$

Since the first terms on the right hand side can be well handled, in the definition of Walsh-Lebesgue points we will consider the second terms, only. We can prove (see Schipp, Wade, Simon and Pál [17] or Weisz [11]) that $\lim_{n \to \infty} \mathbf{d}_n F(x) = 0$ a.e. Since $2^n \int_{I_n(x)} f = E_n f(x)$, by the corresponding martingale theorem $\lim_{n \to \infty} E_n f(x) = f(x)$ a.e. Thus

$$
\frac{1}{2} \sum_{k=0}^{n} 2^k \int_{I_n(x + e_k)} f = \frac{1}{2} \int_{I_n(x)} f - \mathbf{d}_n F(x)
$$

tends to $f(x)$ for a.e. $x \in [0, 1)$ as $n \to \infty$.

Motivated by this fact, the author introduced the one-dimensional Walsh-Lebesgue points in [5] as follows: $x \in [0, 1)$ is a Walsh-Lebesgue point of $f \in L^1[0, 1)$, if

$$
\lim_{n \to \infty} \sum_{k=0}^{n} 2^k \int_{I_n(x + e_k)} |f(t) - f(x)| \, dt = 0.
$$

We proved in [5] that a.e. point $x \in [0, 1)$ is a Walsh-Lebesgue point of an integrable function $f$.

In the multi-dimensional case a point $x \in [0, 1)^d$ is a Walsh-Lebesgue point of $f \in L^1[0, 1)^d$, if

$$
\lim_{n \to \infty} \sum_{j=0}^{d} \sum_{k=0}^{n_j} 2^k \int_{I_n(x + e_k)} |f(t) - f(x)| \, dt = 0,
$$

(4)
where \( 2^k := 2^{k_1} \cdots 2^{k_d} \) and \( e_k := (e_{k_1}, \ldots, e_{k_d}) \). If we define

\[
V_n f(x) := \sum_{j=1}^{d} \sum_{k_j=0}^{n_j} 2^{k_j-n} E_n f(x+e_k),
\]

then it is easy to see that \( x \) is a Walsh-Lebesgue point of \( f \) if and only if

\[
\lim_{n \to \infty} V_n (|f - f(x)|)(x) = 0,
\]

because \( E_n f(x) = 2^n \int_{I_n(x)} f \). We ([18]) have shown the next theorem for the operator

\[
V f := \sup_{n \in \mathbb{N}^d} |V_n f|.
\]

**Theorem 3** For all \( 1 < p \leq \infty \)

\[
\|V f\|_p \leq C_p \|f\|_p \quad (f \in L_p[0,1]^d)
\]

and

\[
\sup_{\rho > 0} \rho \lambda (V f > \rho) \leq C \|f\|_{L_1(\log L)^d} \quad (f \in L_1(\log L)^d[0,1]^d). \quad (5)
\]

It is easy to show that (4) holds for every Walsh polynomials and \( x \in [0,1]^d \). Since the Walsh polynomials are dense in \( L_1(\log L)^d[0,1]^d \), (5) and the usual density argument (see Marcinkiewicz and Zygmund [19]) imply

**Corollary 1** If \( f \in L_1(\log L)^{d-1}[0,1]^d \) then

\[
\lim_{n \to \infty} \sum_{j=1}^{d} \sum_{k_j=0}^{n_j} 2^{k_j-n} \int_{I_n(x+e_k)} |f(t) - f(x)| dt = 0 \quad a.e. \ x \in [0,1]^d,
\]

thus a.e. point is a Walsh-Lebesgue point of \( f \).

## 5 Fejér Means of Walsh-Fourier Series

The *Rademacher functions* are defined by

\[
r(x) := \begin{cases} 
1, & \text{if } x \in [0, \frac{1}{2}); \\
-1, & \text{if } x \in \left[\frac{1}{2}, 1\right), 
\end{cases}
\]
and 
\[ r_n(x) := r(2^n x) \quad (x \in [0, 1), n \in \mathbb{N}). \]

The product system generated by the Rademacher functions is the one-dimensional Walsh system:
\[ w_n := \prod_{k=0}^{\infty} r_k^{n_k}, \]
where \( n = \sum_{k=0}^{\infty} n_k 2^k, \) (0 ≤ \( n_k < 2 \)).

The Kronecker product \( (w_n, n \in \mathbb{N}^d) \) of \( d \) Walsh systems is said to be the \( d \)-dimensional Walsh system. Thus
\[ w_n(x) := w_{n_1}(x_1) \cdots w_{n_d}(x_d) \]
where \( n = (n_1, \ldots, n_d) \in \mathbb{N}^d, x = (x_1, \ldots, x_d) \in [0, 1)^d \).

The \( n \)th Fourier coefficient and the partial sum of \( f \in L_1[0, 1]^d \) are introduced by
\[ \hat{f}(n) := \int_{[0,1]^d} f w_n \, d\lambda \quad (n \in \mathbb{N}^d) \]
and
\[ s_nf := \sum_{j=1}^{d} \sum_{k=0}^{n_j-1} \hat{f}(k) w_k \quad (n \in \mathbb{N}^d). \]

It is known that \( s_{2^m \cdots 2^n} f = E_n f \) (\( n \in \mathbb{N}^d \)) and
\[ s_{2^m \cdots 2^n} f \rightarrow f \quad \text{in } L_p\text{-norm as } n \rightarrow \infty, \]
if \( f \in L_p[0, 1]^d \) (1 ≤ \( p < \infty \)). If \( p > 1 \) then the convergence holds also a.e. (see e.g. Schipp, Wade, Simon and Pál [17] or Weisz [20]).

The one-dimensional Carleson’s theorem was extended to Walsh-Fourier series by Billard [21] and Sjölin [22]: if \( f \in L_p[0, 1) \) (1 < \( p < \infty \)) then
\[ s_n f \rightarrow f \quad \text{a.e. as } n \rightarrow \infty. \]

The a.e. convergence of \( s_n f \) is not true in the multi-dimensional case (Fefferman [14, 15]), however, the analogue of (3) holds: for \( f \in L_2[0, 1)^d \)
\[ s_{n, \ldots, n} f \rightarrow f \quad \text{a.e. as } n \rightarrow \infty, \]
(Móricz [23] or Schipp, Wade, Simon and Pál [17]). In contrary to the trigonometric case, it is unknown whether this result holds for functions in \( L_p[0, 1)^d, \) 1 < \( p < 2 \).
To obtain convergence results for $L_1[0,1)$ or $L(\log L)^{d-1}[0,1)^d$ functions we introduce the Fejér means of $f$ by

$$\sigma_n f := \frac{1}{\prod_{i=1}^d n_i} \sum_{j=1}^d \sum_{k_j=1}^{n_j} s_{k} f = \sum_{j=1}^d \sum_{k_j=0}^{n_j-1} \left( \prod_{i=1}^d \left( 1 - \frac{k_i}{n_i} \right) \right) \hat{f}(k) w_k.$$

If

$$K_n := \frac{1}{n} \sum_{k=0}^{n-1} \left( 1 - \frac{k}{n} \right) w_k \quad (n \in \mathbb{N})$$

denotes the one-dimensional Fejér kernels, then

$$\sigma_n f(x) = \int_{[0,1)^d} f(t) (K_{n_1}(x_1+t_1) \cdots K_{n_d}(x_d+t_d)) \, dt.$$

The Fejér means of $f$ converge to $f$ a.e. if $f \in L(\log L)^{d-1}[0,1)^d$ (see Fine [6] and Schipp [7] for the one-dimensional case, i.e., for integrable functions and Weisz [11] for the multi-dimensional case). For Vilenkin-Fourier series these results are due to Simon [24]. The next result concerning Walsh-Lebesgue points characterizes the set of convergence and was proved by the author in [5] for one dimension and in [18] for higher dimensions.

**Theorem 4** If $f \in L_1(\log L)^{d-1}[0,1)^d$ then

$$\lim_{n \to \infty} \sigma_n f(x) = f(x)$$

for all Walsh-Lebesgue points of $f$.

Note that the convergence $\lim_{n \to \infty} \sigma_n f = f$ a.e. cannot be extended to all $f \in L_1[0,1)^d$ (see Gát [25, 26]) and so Theorem 4 is not true for all $f \in L_1[0,1)^d$.

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**References**


