ON A SUM INVOLVING
THE PRIME COUNTING FUNCTION \( \pi(x) \)

Aleksandar Ivić

An asymptotic formula for the sum of reciprocals of \( \pi(n) \) is derived, where \( \pi(x) \) is the number of primes not exceeding \( x \). This result improves the previous results of De Koninck-Ivić and L. Panaitopol.

Let, as usual, \( \pi(x) = \sum_{p \leq x} 1 \) denote the number of primes not exceeding \( x \). The prime number theorem (see e.g., [2, Chapter 12]) in its strongest known form states that

\[
\pi(x) = \text{li} x + R(x),
\]

with

\[
\text{li} x := \int_2^x \frac{dt}{\log t} = x \left( \frac{1}{\log x} + \frac{1!}{\log^2 x} + \cdots + \frac{m!}{\log^{m+1} x} + O\left( \frac{1}{\log^{m+2} x} \right) \right)
\]

for any fixed integer \( m \geq 0 \), and

\[
R(x) \ll x \exp\left( -C\delta(x) \right), \quad \delta(x) := (\log x)^{3/5} (\log \log x)^{-1/5} \quad (C > 0),
\]

where henceforth \( C, C_1, \ldots \) will denote absolute constants. In [1, Theorem 9.1] J.-M. De Koninck and the author proved that

\[
\sum_{2 \leq n \leq x} \frac{1}{\pi(n)} = \frac{1}{2} \log^2 x + O(\log x).
\]

Recently L. Panaitopol [1] improved (4) to

\[
\sum_{2 \leq n \leq x} \frac{1}{\pi(n)} = \frac{1}{2} \log^2 x - \log x - \log \log x + O(1).
\]

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One obtains (5) from the asymptotic formula

\[
\frac{1}{\ln x} = \frac{1}{x} \left( \log x - 1 - \frac{k_1}{\log x} - \frac{k_2}{\log^2 x} - \cdots - \frac{k_m (1 + \alpha_m(x))}{\log^m x} \right),
\]

where \( \alpha_m(x) \ll_m 1/\log x \), and the constants \( k_1, \ldots, k_m \) are defined by the recurrence relation

\[
k_{m+1} + k_{m+1} + \cdots + (m-1)! k_1 = m \cdot m! \quad (m \in \mathbb{N}),
\]

so that \( k_1 = 1, k_2 = 3, k_3 = 13, \) etc. This was established in [3]. Using (6) we shall give a further improvement of (4), contained in the following

**Theorem.** For any fixed integer \( m \geq 2 \) we have

\[
\sum_{2 \leq n \leq x} \frac{1}{\pi(n)} = \frac{1}{2} \log^2 x - \log x - \log \log x + C
\]

\[
+ \frac{k_2}{\log x} + \frac{k_3}{2 \log^2 x} + \cdots + \frac{k_m}{(m-1) \log^{m-1} x} + O \left( \frac{1}{\log^m x} \right),
\]

where \( C \) is an absolute constant, and \( k_2, \ldots, k_m \) are the constants defined by (7).

**Proof.** From (1) we have

\[
\sum_{2 \leq n \leq x} \frac{1}{\pi(n)} = 1 + \sum_{3 \leq n \leq x} \frac{1}{\ln n} - \sum_{3 \leq n \leq x} \frac{R(n)}{\ln(n + R(n))}
\]

\[
= \sum_{3 \leq n \leq x} \frac{1}{\ln n} + \left( 1 - \sum_{n=3}^{\infty} \frac{R(n)}{\ln(n + R(n))} \right) + \sum_{n > x} \frac{R(n)}{\ln(n + R(n))}
\]

\[
= \sum_1 + C_1 + \sum_2,
\]

say. By using the bound \( \ln x \ll x/\log x \) and (3) it is seen that

\[
\sum_2 = \sum_{n > x} \frac{R(n)}{\ln(n + R(n))} \ll \sum_{n > x} \frac{1}{n} e^{-C\delta(n)/2}
\]

\[
\ll e^{-C\delta(x)/3} \int_{x-1}^{\infty} \frac{1}{t} e^{-C\delta(t)/6} dt \ll e^{-C\delta(x)/3},
\]

since \( \delta(x) \) is increasing for \( x \geq x_0 \), and the substitution \( \log t = u \) easily shows that the above integral is convergent. To evaluate \( \sum_1 \) we need the familiar EULER–MACLAURIN summation formula (see e.g., [2, eq. (A.23)]) in the form

\[
\sum_{x < n \leq X} f(n) = \int_X^Y f(t) dt - \psi(Y)f(Y) + \psi(X)f(X) + \int_X^Y \psi(t)f'(t) dt,
\]
On a sum involving the prime counting function $\pi(x)$

where $\psi(x) = x - \lfloor x \rfloor - 1/2$ and $f(x) \in C^1[X, Y]$. We obtain from (6), for $m \geq 2$ a fixed integer,

$$
\sum_1 = \sum_{3 \leq n \leq x} \frac{1}{n} \\
= \sum_{3 \leq n \leq x} \frac{1}{n} \left( \log n - 1 - \frac{k_1}{\log n} - \frac{k_2}{\log^2 n} - \ldots - \frac{k_m(1 + \alpha_m(n))}{\log^m n} \right),
$$

and we evaluate each sum in (10) by using (9). We obtain

$$
\sum_{3 \leq n \leq x} \frac{\log n}{n} = \frac{1}{2} \log^2 x + c_1 + O\left(\frac{\log x}{x}\right),
$$

$$
\sum_{3 \leq n \leq x} \frac{1}{n} = \log x + c_2 + O\left(\frac{1}{x}\right),
$$

$$
\sum_{3 \leq n \leq x} \frac{k_1}{\log n} = \log \log x + c_3 + O\left(\frac{1}{x \log x}\right),
$$

and for $2 \leq r \leq m$

$$
\sum_{3 \leq n \leq x} \frac{k_r}{n \log^{r-1} n} = k_r \int_3^x \frac{dt}{t \log^r t} + C_r + O\left(\frac{1}{x \log^r x}\right)
$$

$$
= k_r \int_3^{\infty} \frac{dt}{t \log^r t} - k_r \int_x^{\infty} \frac{dt}{t \log^r t} + C_r + O\left(\frac{1}{x \log^r x}\right)
$$

$$
= D_r - \frac{k_r}{(r-1) \log^{r-1} x} + O\left(\frac{1}{x \log^r x}\right)
$$

with

$$
D_r = C_r + k_r \int_3^{\infty} \frac{dt}{t \log^r t}.
$$

Finally in view of $\alpha_m(x) \ll 1/\log x$ it follows that, for $m \geq 2$ fixed,

$$
\sum_{3 \leq n \leq x} \frac{k_m \alpha_m(n)}{n \log^m n} = \sum_{n=3}^{\infty} \frac{k_m \alpha_m(n)}{n \log^m n} + O\left(\frac{1}{\log^m x}\right).
$$

Putting together the above expressions in (10) we infer that

$$
\sum_1 = \frac{1}{2} \log^2 x - \log x - \log \log x + C
$$

$$
+ \frac{k_2}{\log x} + \frac{k_3}{2 \log^2 x} + \ldots + \frac{k_m}{(m-1) \log^{m-1} x} + O\left(\frac{1}{\log^m x}\right),
$$

and then (8) easily follows with

$$
C = C_1 + c_1 - c_2 - c_3 - D_2 - \ldots - D_m - \sum_{n=3}^{\infty} \frac{k_m \alpha_m(n)}{n \log^m n}.$$
The constant $C$ in (8) does not depend on $m$, which can be easily seen by taking two different values of $m$ and then comparing the results.

Note that we can evaluate directly $\sum_1$ by the Euler-Maclaurin summation formula to obtain

$$\sum_1 = \int_3^x \frac{dt}{\ln t} + C_0 + O\left(\frac{\log x}{x}\right).$$

Integration by parts gives, for $x > 3$,

$$\int_3^x \frac{dt}{\ln t} = \int_3^x \log t \, d(\log \ln t) = \log x \log (\ln x) - \int_3^x \frac{\log (\ln t)}{t} \, dt - \log 3 \log \ln 3,$$

which inserted into (11) gives another expression for our sum, namely

$$\sum_{2 \leq n \leq x} \frac{1}{\pi(n)} = \log x \log (\ln x) - \int_3^x \frac{\log (\ln t)}{t} \, dt + B + O\left(e^{-D\delta(x)}\right) \quad (D > 0),$$

from which we can again deduce (8) by using (2). The advantage of (12) is that it has a sharper error term than (8), but on the other hand the expressions on the right-hand side of (12) involve the non-elementary function $\ln x$. Note also that the Riemann hypothesis (that all complex zeros of the Riemann zeta-function $\zeta(s)$ have real parts equal to $\frac{1}{2}$) is equivalent to the statement (see [2]) that, for any given $\varepsilon > 0$, $R(x) \ll x^{1/2+\varepsilon}$ in (3), which would correspondingly improve the error term in (12) to $O\left(x^{-1/2+\varepsilon}\right)$.

REFERENCES


Katedra Matematike RGF-a, 
Univerzitet u Beogradu, 
Đušina 7, 11000 Beograd, 
Serbia (Yugoslavia).
E–mail: eivica@ubbg.etf.bg.ac.yu, aivic@rgf.bg.ac.yu

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