INTEGRAL COMPLETE SPLIT GRAPHS

Pierre Hansen, Hadrien Mélot, Dragan Stevanović

We give characterizations of integral graphs in the family of complete split graphs and a few related families of graphs.

1. INTRODUCTION

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be undirected graphs without loops or multiple edges. The union $G_1 \cup G_2$ of graphs $G_1$ and $G_2$ is the graph $G = (V, E)$ for which $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. The notation $nG$ is short for $G \cup G \cup \ldots \cup G$.

The complete product $G_1 \nabla G_2$ of graphs $G_1$ and $G_2$ is the graph obtained from $G_1 \cup G_2$ by joining every vertex of $G_1$ with every vertex of $G_2$. The sum $G_1 + G_2$ of graphs $G_1$ and $G_2$ is the graph with the vertex set $V(G_1) \times V(G_2)$ in which two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent if and only if $u_1 = v_1$ and $(u_2, v_2) \in E_2$ or $u_2 = v_2$ and $(u_1, v_1) \in E_1$. Further, let $K_n$ denote the complete graph on $n$ vertices, and let $K_n$ denote the graph with $n$ vertices and no edges.

For $a, b, n \in \mathbb{N}$ we define the following classes of graphs:

- the complete split graph $CS_b^a \cong K_a \nabla K_b$;
- the multiple complete split-like graph $MCS_{b,n}^a \cong K_a \nabla (nK_b)$;
- the extended complete split-like graph $ECS_b^a \cong K_a \nabla (K_b + K_2)$;
- the multiple extended complete split-like graph $MECS_{b,n}^a \cong K_a \nabla (n(K_b + K_2))$.

2000 Mathematics Subject Classification: 05C50
Keywords and Phrases: integral graphs, split graphs
A graph is called integral if all the eigenvalues of its adjacency matrix are integers. The search for integral graphs, initiated by F. Harary and A. Schwenk in [5] and continued in many papers thereafter, revealed that not only the number of integral graphs is infinite, but that one can find them in almost all classes of graphs. For a recent survey on integral graphs, see [2].

All 263 integral graphs with up to 11 vertices were enumerated in [1] (see also [2]). A close look at graphical representations of these graphs showed that among several known families, some of them were complete split graphs and some had split graphs as induced subgraphs (see Fig. 0.1).

However, not all complete split graphs are integral. Computer investigations with Matlab [6] on complete split graphs with up to 500 vertices led to the conjecture that the following two families of complete split graphs are integral:

(i) Complete split graphs $CS^n_k$ satisfying

$$a = \frac{i^2}{2} + (b - 1) \left\lfloor \frac{i}{2} \right\rfloor \left\lceil \frac{i + 2}{2} \right\rceil \quad (i \in \mathbb{N}).$$

Moreover, if $b$ is the power of a prime, there are no other integral complete split graphs.

(ii) Complete split graphs $CS^n_k$ satisfying $b = 4k + 2$ for $k \in \mathbb{N}$ and

$$a = -\frac{b}{4} + (b - 1) \left\lfloor \frac{i}{2} \right\rfloor + \left\lceil \frac{i}{2} \right\rceil \left\lceil \frac{i + 2}{2} \right\rceil \quad (i \in \mathbb{N}).$$

In the next section we give a necessary and sufficient condition for the complete product of two regular graphs to be integral. Using this condition in Section 3 we characterize integral graphs in the families of complete split graphs and multiple complete split-like graphs, by giving the explicit formulae for their parameters $a$, $b$ and $n$. We also show that the above families (1) and (2) satisfy the corresponding formula. At the end, in Section 4 we characterize integral graphs in the families of extended complete split-like graphs and multiple extended complete split-like graphs.

Figure 0.1: Examples of graphs.
2. INTEGRALITY CONDITION FOR THE COMPLETE PRODUCT OF GRAPHS

The following theorem found in [3] is proven in [4].

**Theorem 1** (Finck, Grohmann). For \(i = 1, 2\) let \(G_i\) be regular graphs of degree \(r_i\) with \(n_i\) vertices. The characteristic polynomial of the complete product of graphs \(G_1\) and \(G_2\) is given by the relation

\[
P_{G_1 \square G_2}(\lambda) = \frac{P_{G_1}(\lambda)P_{G_2}(\lambda)}{\lambda - r_1(\lambda - r_2) - n_1n_2}. \tag{1}
\]

Since \(G_i\) is regular graph, its largest eigenvalue is equal to \(r_i\) (with the eigenvector equal to all-1 vector) and the fraction \(P_{G_i}(\lambda)/(\lambda - r_i)\) is the polynomial with roots equal to the remaining eigenvalues of \(G_i\) with the same multiplicities. Thus in order that the complete product \(G_1 \square G_2\) of two regular graphs is integral both \(G_1\) and \(G_2\) must be integral and the expression

\[
Q(\lambda) = (\lambda - r_1)(\lambda - r_2) - n_1n_2
\]

must have integer roots. The roots of \(Q(\lambda)\) are equal to

\[
\lambda_{1,2} = \frac{r_1 + r_2 \pm \sqrt{(r_1 - r_2)^2 + 4n_1n_2}}{2},
\]

and they are integers if and only if \(r_1 + r_2\) and \(\sqrt{(r_1 - r_2)^2 + 4n_1n_2}\) are integers of the same parity. The last fact means that there exists \(k \in \mathbb{N}\) such that \((r_1 - r_2)^2 + 4n_1n_2 = (|r_1 - r_2| + 2k)^2\), wherefrom we get the following integrality condition

\[
n_1n_2 = k(k + |r_1 - r_2|). \tag{3}
\]

Thus we have proved

**Corollary 1.** For \(i = 1, 2\) let \(G_i\) be regular graphs of degree \(r_i\) with \(n_i\) vertices. The complete product \(G_1 \square G_2\) is integral graph if and only if both \(G_1\) and \(G_2\) are integral graphs and there exists \(k \in \mathbb{N}\) such that the integrality condition (3) holds.

3. INTEGRAL COMPLETE SPLIT GRAPHS

We have that \(CS_b^a \cong \overline{K}_a \square K_b\). Graph \(\overline{K}_a\) is regular with degree 0 (\(n_1 = a, r_1 = 0\)), while \(K_b\) is regular with degree \(b - 1\) (\(n_2 = b, r_2 = b - 1\)). By Corollary 1, graph \(CS_b^a\) is integral if and only if there exists \(k \in \mathbb{N}\) such that \(ab = k(k + b - 1)\), hence

\[
a = k + \frac{k(k - 1)}{b} \quad \text{and} \quad b \text{ divides } k(k - 1). \tag{4}
\]
Since the greatest common divisor \((k, k - 1)\) of \(k\) and \(k - 1\) is equal to 1, we have that
\[
b = (b, k(k - 1)) = (b, k) \cdot (b, k - 1).
\]
Let \(p = (b, k)\) and \(q = (b, k - 1)\). Thus \(b = pq\), \((p, q) = 1\), \(p|k\) and \(q|k - 1\). Let \(\alpha, \beta \in \mathbb{Z}\) be determined by the Euclidean algorithm such that \(p\alpha - q\beta = 1\), and let \(k' = k - p\alpha = k - 1 - q\beta\). From \(p|k\) it follows that \(p|k'\), while from \(q|k - 1\) it follows that \(q|k'\). Since \((p, q) = 1\) we have that \(pq|k'\) and
\[
k = pq + cpq \quad \text{for some } c \in \mathbb{Z},
\]
while \(k - 1 = q\beta + cpq\). Now from (4) it follows that
\[
a = pq + c(q + cp)(\beta + cp) = (\alpha + c)(\beta + p + cp).
\]

Thus we have proved

**Theorem 2.** The complete split graph \(CS_a^b\) is integral if and only if there exist \(p, q \in \mathbb{N}\) with \((p, q) = 1\) and \(c \in \mathbb{Z}\) such that
\[
\alpha + cq > 0, \quad a = (\alpha + cq)(\beta + cp), \quad \text{and} \quad b = pq,
\]
where \(\alpha, \beta \in \mathbb{Z}\) are determined by the Euclidean algorithm such that \(p\alpha - q\beta = 1\).

Let us return now to the conjectured families of integral complete split graphs.

The family (1) for odd \(i\) corresponds to the case \(p = 1\), \(q = b\), \(\alpha = 1\), \(\beta = 0\) and \(c = \left\lfloor \frac{i}{2} \right\rfloor\), while for even \(i\) it corresponds to the case \(p = b\), \(q = 1\), \(\alpha = 0\), \(\beta = -1\) and \(c = \left\lfloor \frac{i}{2} \right\rfloor\).

Moreover, if \(b\) is a power of prime, we have only two possibilities. The first possibility is that \(p = b\) and \(q = 1\), and thus \(\alpha = 1\) and \(\beta = (b - 1)\). In this case we have that
\[
a = (1 + c)(b - 1 + b + cb) = (c + 1)^2 + (c + 1)(c + 2)(b - 1),
\]
which corresponds to \(i = 2c + 2\). The second possibility is that \(p = 1\) and \(q = b\), wherefrom \(\alpha = b + 1\) and \(\beta = 1\). In this case we have that
\[
a = (b + 1 + cb)(1 + 1 + c) = (c + 2)^2 + (c + 1)(c + 2)(b - 1),
\]
which corresponds to \(i = 2c + 3\).

The family (2) for odd \(i\) corresponds to the case \(p = 2k + 1\), \(q = 2\), \(\alpha = 1\), \(\beta = k\) and \(c = \left\lfloor \frac{i}{2} \right\rfloor\), while for even \(i\) it corresponds to the case \(p = 2\), \(q = 2k + 1\), \(\alpha = k + 1\), \(\beta = 1\) and \(c = \left\lfloor \frac{i}{2} \right\rfloor - 1\).

Multiple complete split-like graphs provide a simple generalization of complete split graphs. We have that \(MCS_{a,b}^n \cong \overline{K}_a \nabla (nK_b)\). Graph \(\overline{K}_a\) is regular with degree 0 (\(n_1 = a, r_1 = 0\)), while \(nK_b\) is regular with degree \(b - 1\) (\(n_2 = nb, r_2 = b - 1\)).
By Corollary 1, graph $MCS_{b,n}^a$ is integral if and only if there exists $k \in \mathbb{N}$ such that $anb = k(k + b - 1)$, hence

$$an = k + \frac{k(k - 1)}{b} \quad \text{and} \quad b \text{ divides } k(k - 1).$$

Repeating the consideration from above we get the following

**Theorem 3.** The multiple complete split graph $MCS_{b,n}^a$ is integral if and only if there exist $p, q \in \mathbb{N}$ with $(p, q) = 1$ and $c \in \mathbb{Z}$ such that

$$\alpha + cq > 0, \quad an = (\alpha + cq)(\beta + p + cp), \quad \text{and} \quad b = pq,$$

where $\alpha, \beta \in \mathbb{Z}$ are determined by the Euclidean algorithm such that $p\alpha - q\beta = 1$.

\[\blacksquare\]

### 4. INTEGRAL EXTENDED COMPLETE SPLIT-LIKE GRAPHS

We have that $ECS_b^a \cong K_a \nabla (K_b + K_2)$. Graph $K_a$ is regular with degree 0 ($n_1 = a, r_1 = 0$), while $K_b + K_2$ is regular with degree $b$ ($n_2 = 2b, r_2 = b$). By Corollary 1, graph $ECS_b^a$ is integral if and only if there exists $k \in \mathbb{N}$ such that $2ab = k(k + b)$. From this it follows that

$$(5) \quad a = \frac{1}{2} \left( k + \frac{k^2}{b} \right) \quad \text{and} \quad 2b \text{ divides } k^2.$$

Let

$$b = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$$

be a prime factorization of $b$, such that $p_1 \leq p_2 \leq \ldots \leq p_r$. From $b|k^2$ it follows that $p_i^{\lfloor \frac{\alpha_i}{2} \rfloor} | k$ for each $i = 1, 2, \ldots, r$, and we can write

$$k = p_1^{\lfloor \frac{\alpha_1}{2} \rfloor} p_2^{\lfloor \frac{\alpha_2}{2} \rfloor} \cdots p_r^{\lfloor \frac{\alpha_r}{2} \rfloor} \cdot c, \quad c \in \mathbb{N}.$$

Let odd $(n), n \in \mathbb{N}$, be the characteristic function of the set of odd numbers, i.e.

$$\text{odd}(n) = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases} = 2 \left\lfloor \frac{n}{2} \right\rfloor - n.$$

From (5) it follows that

$$(6) \quad a = \frac{1}{2} \cdot c \cdot \left( p_1^{\lfloor \frac{\alpha_1}{2} \rfloor} p_2^{\lfloor \frac{\alpha_2}{2} \rfloor} \cdots p_r^{\lfloor \frac{\alpha_r}{2} \rfloor} + p_1^{\text{odd}(\alpha_1)} p_2^{\text{odd}(\alpha_2)} \cdots p_r^{\text{odd}(\alpha_r)} \cdot c \right).$$

We see that $a \in \mathbb{N}$ if either $c$ is even or $b$ is odd. In this second case, all primes $p_1, p_2, \ldots, p_r$ are odd and the right hand side of (6) contains the product of two numbers of distinct parities, which is always even.
In the remaining case when both $c$ is odd and $b$ is even, it must hold that

$$2 | p_1^{\lceil \frac{a_1}{2} \rceil} p_2^{\lceil \frac{a_2}{2} \rceil} \cdots p_r^{\lceil \frac{a_r}{2} \rceil} + p_1^{\text{odd}(a_1)} p_2^{\text{odd}(a_2)} \cdots p_r^{\text{odd}(a_r)} \cdot c.$$ 

Since $b$ is even, we have that $p_1 = 2$ and $\lceil \frac{a_1}{2} \rceil \geq 1$, so that the above condition, taking into account that $p_2, \ldots, p_r$ are all odd, becomes $2 | p_1^{\text{odd}(a_1)}$, which is equivalent to the condition that $a_1$ is odd.

We summarize this in the following

**Theorem 4.** Let $b = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ be a prime factorization of $b$. The extended complete split graph $ECS_b$ is integral if and only if there exists $c \in \mathbb{N}$ such that

$$a = \frac{1}{2} \cdot c \cdot \left( p_1^{\lceil \frac{a_1}{2} \rceil} p_2^{\lceil \frac{a_2}{2} \rceil} \cdots p_r^{\lceil \frac{a_r}{2} \rceil} + p_1^{\text{odd}(a_1)} p_2^{\text{odd}(a_2)} \cdots p_r^{\text{odd}(a_r)} \cdot c \right),$$

and either $c$ is even or $b$ is odd or the highest power of 2 which divides $b$ is odd. $\blacksquare$

Similar as with complete split graphs, multiple extended complete split-like graphs provide a simple generalization of extended complete split-like graphs. We have that $MECS_{b,n}^a \cong K_a \nabla \left(n(K_b + K_2)\right)$. Graph $K_a$ is regular with degree 0 $(n_1 = a, r_1 = 0)$, while $n(K_b + K_2)$ is regular with degree $b (n_2 = 2ab, r_2 = b)$. By Corollary 1, graph $MECS_{b,n}^a$ is integral if and only if there exists $k \in \mathbb{N}$ such that $2anb = k(k+b)$. From this it follows that

$$an = \frac{1}{2} \left(k + \frac{k^2}{b}\right) \quad \text{and} \quad 2b \text{ divides } k^2.$$ 

Repeating the consideration from above we get the following

**Theorem 5.** Let $b = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ be a prime factorization of $b$. The multiple extended complete split graph $MECS_{b,n}^a$ is integral if and only if there exists $c \in \mathbb{N}$ such that

$$an = \frac{1}{2} \cdot c \cdot \left( p_1^{\lceil \frac{a_1}{2} \rceil} p_2^{\lceil \frac{a_2}{2} \rceil} \cdots p_r^{\lceil \frac{a_r}{2} \rceil} + p_1^{\text{odd}(a_1)} p_2^{\text{odd}(a_2)} \cdots p_r^{\text{odd}(a_r)} \cdot c \right),$$

and either $c$ is even or $b$ is odd or the highest power of 2 which divides $b$ is odd. $\blacksquare$

**Remark.** Consider the even more general case of graph $K_a \nabla (G + K_n)$, where $G$ is an integral, regular graph with $m$ vertices and degree $r$. Here we have that $n_1 = a, r_1 = 0$ and $n_2 = mn, r_2 = r + n - 1$, and the integrality condition (3) says that $K_a \nabla (G + K_n)$ is an integral graph if and only if there exists $k \in \mathbb{N}$ such that

$$anm = k(k + r + n - 1).$$

If $a = mn + r + n - 1$ then $k = mn$ satisfies the above condition, but the task of characterization of set of parameters $a, m, r, n$ for which the graph $K_a \nabla (G + K_n)$ is integral appears to be difficult.
REFERENCES


