ANTIREGULAR GRAPHS ARE UNIVERSAL FOR TREES

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A graph on \( n \) vertices is antiregular if its vertex degrees take on \( n - 1 \) different values. For every \( n \geq 2 \) there is a unique connected antiregular graph on \( n \) vertices. Call it \( A_n \). (The unique disconnected antiregular graph on \( n \) vertices is \( A_n' \).) The main result of this note is that every tree on \( n \) vertices is isomorphic to a subgraph of \( A_n \).

1. ANTIREGULAR GRAPHS

Let \( G = (V, E) \) be a graph with vertex set \( V = V(G) = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E \). Denote by \( d_G(v) \) the degree of vertex \( v \), so that \( n - 1 \geq d_G(v) \geq 0 \). If \( d_G(v_1) = d_G(v_2) = \cdots = d_G(v_n) \), then \( G \) is regular. At the other extreme are graphs whose vertex degrees are as different from each other as possible.

If \( n \geq 2 \), then vertex \( v \) has degree \( n - 1 \) if and only if it is a dominating vertex, adjacent to every other vertex, which precludes the existence of an isolated vertex of degree 0. Since no graph can have both a dominating vertex and an isolated vertex, some two vertices of \( G \) have the same degree. Following [11], we say that \( G \) is antiregular if its vertex degrees attain \( n - 1 \) different values, and adopt the convention that \( K_1 \), the (unique) graph on 1 vertex, is antiregular.

Let \( d(G) = (d_1, d_2, \ldots, d_n) \) be the vertex degrees of \( G \) arranged in non-increasing order, \( d_1 \geq d_2 \geq \cdots \geq d_n \). Because \( d(G^c) = (n - 1 - d_n, n - 1 - d_{n - 1}, \ldots, n - 1 - d_1) \), \( G \) is antiregular if and only if its complement is antiregular. Moreover, \( G \) has a dominating vertex if and only if \( G^c \) has an isolated vertex. Apart from \( K_1 \), antiregular graphs come in natural pairs, one of which is connected and the other of which is not.

Theorem 1. [1] Suppose \( n \geq 2 \). Then, up to isomorphism, there is a unique connected antiregular graph on \( n \) vertices, and its repeated vertex degree is \( \lfloor n/2 \rfloor \).

Proof sketch. The unique connected graph on 2 vertices is the complete graph \( K_2 \) having two vertices of degree 1. Let \( G \) is a connected antiregular graph on \( n \geq 2 \).

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vertices. Then \( d(G) = (n-1, n-2, \ldots) \). If \( d_G(u) = n-1 \) and \( d_G(w) = n-2 \), then \( G - u \) is an antiregular graph on \( n-1 \) vertices which is connected because \( w \) is a dominating vertex. The result follows by induction.

**Definition.** Define by \( A_n \) the unique connected antiregular graph on \( n \) vertices.

\[
A_2 \quad A_3 \quad A_4 \quad A_5
\]

Figure 1

2. UNIVERSAL GRAPHS

Graph \( G \) on \( n \) vertices is *universal for trees* if every tree on \( n \) vertices is isomorphic to a subgraph of \( G \). (See, e.g., [2]–[7], [9]–[10], [12], and [14]–[15].)

**Theorem 2.** The connected antiregular graph \( A_n \) is universal for trees.

**Proof.** Recall that a forest is a graph without cycles, i.e., a graph of whose connected components is a tree. We will prove the theorem by showing that every forest on \( n \) vertices is isomorphic to a subgraph of \( A_n \).

If \( G = (V, E) \) and \( H = (W, F) \) are graphs on disjoint sets of vertices \( V \) and \( W \), their *union* is \( G + H = (V \cup W, E \cup F) \). The *join* of \( G \) and \( H \) is \( G \lor H = (G^c + H^c)^c \), the graph obtained from \( G + H \) by adding new edges joining each vertex of \( G \) to every vertex of \( H \).

Because \( A_1 = K_1 \) and \( A_2 = K_2 \), every graph on \( n \) vertices is isomorphic to a subgraph of \( A_n \), \( n \leq 2 \). So, suppose \( n \geq 3 \). Because \( A_n + K_1 \) is a disconnected antiregular graph on \( n + 1 \) vertices, it must be the complement of \( A_{n+1} \), i.e.,

\[
A_{n+1} = (A_n + K)^c = A_n^c \lor K_1 = (A_{n-1} + K_1) \lor K_1.
\]

Let \( F \) be a forest on \( n + 1 \) vertices. Suppose \( u \) is a pendant (degree 1) vertex of \( F \) with unique neighbour \( v \). Then \( F' = F - u - v \) is a forest on \( n - 1 \) vertices which, by induction, is isomorphic to a subgraph of \( A_{n-1} \). Because it is isomorphic to a subgraph of the tree \( (F' + u) \lor v \), \( F \) is isomorphic to a subgraph of \( (A_{n-1} + u) \lor v = A_{n+1} \). □

3. CONCLUDING REMARKS

Antiregular graphs have many other interesting properties. They are, for example, *threshold graphs*. (See, e.g., [13].) If \( G \) is a threshold graph, then both \( G \) and \( G^c \) are *chordal* [8]. Thus, \( A_n \) is a *perfect* graph. Its line graph is hamiltonian. Its chromatic and matching numbers are \( \chi(A_n) = \lfloor n/2 \rfloor + 1 \) and \( \mu(A_n) = \lfloor n/2 \rfloor \),
respectively. If $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ are the eigenvalues of its adjacency matrix, then either $\gamma_r = 0 = \gamma_{n-r+1}$, or they have opposite signs, $1 \leq r \leq n$, i.e., while $A_n$ is not bipartite (for $n \geq 4$) it has bipartite character. Finally, the Laplacian eigenvalues of $A_n$ consist of all but one of the integers $0, 1, 2, \ldots, n$. The “missing eigenvalue” is $\lambda = \lfloor (n+1)/2 \rfloor$.

REFERENCES