SOME IDENTITIES FOR THE RIEMANN ZETA-FUNCTION

Aleksandar Ivić

Several identities for the Riemann zeta-function $\zeta(s)$ are proved. For example, if $s = \sigma + it$ and $\sigma > 0$, then

$$\int_{-\infty}^{\infty} \left| \frac{1 - 2^{1-s}}{s} \right|^2 \zeta(s) \, dt = \frac{\pi}{\sigma} (1 - 2^{1-2\sigma}) \zeta(2\sigma).$$

Let as usual $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ ($\Re s > 1$) denote the Riemann zeta-function. The motivation for this note is the quest to evaluate explicitly integrals of $|\zeta(\frac{1}{2} + it)|^{2k}$, $k \in \mathbb{N}$, weighted by suitable functions. In particular, the problem is to evaluate in closed form

$$\int_{0}^{\infty} \left( 3 - \sqrt{8} \cos(t \log 2) \right)^k |\zeta(\frac{1}{2} + it)|^{2k} \frac{dt}{(\frac{1}{4} + t^2)^k} \quad (k \in \mathbb{N}).$$

When $k = 1, 2$ this may be done, thanks to the identities which will be established below. The first identity in question is given by

**Theorem 1.** Let $s = \sigma + it$. Then for $\sigma > 0$ we have

$$(1) \quad \int_{-\infty}^{\infty} \left| \frac{1 - 2^{1-s}}{s} \right|^2 \zeta(s) \, dt = \frac{\pi}{\sigma} (1 - 2^{1-2\sigma}) \zeta(2\sigma).$$

Since $\lim_{s \to 1} (s - 1) \zeta(s) = 1$, then setting in (1) $\sigma = \frac{1}{2}$ we obtain the following

**Corollary 1.**

$$(2) \quad \int_{0}^{\infty} \left( 3 - \sqrt{8} \cos(t \log 2) \right)|\zeta(\frac{1}{2} + it)|^2 \frac{dt}{\frac{1}{4} + t^2} = \pi \log 2.$$
Another identity, which relates directly the square of \( \zeta(s) \) to a Mellin-type integral, is contained in

**Theorem 2.** Let \( \chi_A(x) \) denote the characteristic function of the set \( A \), and let

\[
\varphi(x) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{1}^{x} \chi_{[2m-1, 2m)} \left( \frac{x}{u} \right) \chi_{[2n-1, 2n)}(u) \frac{du}{u} \quad (x \geq 1).
\]

Then for \( \sigma > 0 \) we have

\[
s^2 \int_{1}^{\infty} \varphi(x)x^{-s-1} \, dx = (1 - 2^{1-s})^2 \zeta^2(s).
\]

From (4) we obtain the following

**Corollary 2.**

\[
\int_{0}^{\infty} s^2 \left( 3 - \sqrt{8 \cos(t \log 2)} \right)^2 \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 \frac{dt}{(4 + t^2)^2} = \pi \int_{1}^{\infty} \varphi^2(x) \frac{dx}{x^2}.
\]

The integral on the right-hand side of (5) is elementary, but nevertheless its evaluation in closed form is complicated.

**Proof of Theorem 1.** We start from (see e.g., [1, Chapter 1]) the identity

\[
(1 - 2^{1-s}) \zeta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} \quad (\sigma > 0)
\]

and

\[
\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{\sigma^2 + x^2} \, dx = \frac{\pi e^{-|\alpha| \sigma}}{\sigma} \quad (\alpha \in \mathbb{R}, \sigma > 0),
\]

which follows by the residue theorem on integrating \( e^{\alpha z}/(\sigma^2 + z^2) \) over the contour consisting of \([-R, R]\) and semicircle \( |z| = R, 3m z > 0 \) and letting \( R \to \infty \). By using (6) and (7) it is seen that the left-hand side of (1) becomes

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} (mn)^{-\sigma} \int_{-\infty}^{\infty} \left( \frac{m}{n} \right)^{it} \frac{dt}{\sigma^2 + t^2}
= \frac{\pi}{\sigma} \left( \zeta(2\sigma) + 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} (mn)^{-\sigma} \int_{-\infty}^{\infty} \frac{\cos(t \log m)}{\sigma^2 + t^2} \, dt \right)
= \frac{\pi}{\sigma} \left( \zeta(2\sigma) + 2 \sum_{m=1}^{\infty} (-1)^{m-\sigma} \sum_{n=1}^{m-1} (-1)^n \cdot e^{-\sigma \log m} \right)
= \frac{\pi}{\sigma} \left( \zeta(2\sigma) + 2 \sum_{m=1}^{\infty} (-1)^{m-\sigma} \sum_{n=1}^{m-1} (-1)^n \right)
= \frac{\pi}{\sigma} \left( \zeta(2\sigma) + 2 \sum_{k=1}^{\infty} (-1)^{2k} (2k)^{-2\sigma} (-1) \right) = \frac{\pi}{\sigma} (1 - 2^{1-2\sigma}) \zeta(2\sigma).
\]
This holds initially for $\sigma > 1$, but by analytic continuation it holds for $\sigma > 0$ as well.

We shall provide now a second proof of Theorem 1. As in the formulation of Theorem 2, let $\chi_A(x)$ denote the characteristic function of the set $A$, and let the interval $[a, b)$ denote the set of numbers $\{x : a \leq x < b\}$. Then, for $\sigma > 0$, we have

$$\int_1^\infty x^{-s-1} \sum_{n=1}^\infty \chi_{[2n-1, 2n]}(x) \, dx = \sum_{n=1}^{2n-1} x^{-s-1} \, dx$$

$$= \frac{1}{s} \sum_{n=1}^\infty \left[(2n-1)^{-s} - (2n)^{-s}\right] = \frac{(1 - 2^{1-s}) \zeta(s)}{s}$$

in view of (6). Now we invoke Parseval’s identity for Mellin transforms (see e.g., [1] and [3]). We need this identity for the modified Mellin transforms, defined by

$$F^*(s) \equiv m[f(x)] := \int_1^\infty f(x) x^{-s-1} \, dx.$$  

The properties of this transform were developed by the author in [2]. In particular, we need Lemma 3 of [2] which says that

$$\int_1^\infty f(x) g(x) x^{1-2\sigma} \, dx = \frac{1}{2\pi i} \int_{\Re s = \sigma} F^*(s) \overline{G^*(s)} \, ds$$

if $F^*(s) = m[f(x)]$, $G^*(s) = m[g(x)]$, and $f(x), g(x)$ are real-valued, continuous functions for $x > 1$, such that

$$x^{\frac{1}{2}-\sigma} f(x) \in L^2(1, \infty), \quad x^{\frac{1}{2}-\sigma} g(x) \in L^2(1, \infty).$$

From (8) and (9) we obtain, for $\sigma > 0$,

$$\int_1^\infty \frac{1}{x^2} \left( \sum_{n=1}^\infty \chi_{[2n-1, 2n]}(x) \right)^2 x^{1-2\sigma} \, dx = \frac{1}{2\pi i} \int_{\Re s = \sigma} \left| \frac{(1 - 2^{1-s}) \zeta(s)}{s} \right|^2 \, ds.$$  

But as $\chi_A(x) = \chi_A(x)$, it is easily found that the left-hand side of the above identity equals

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \int_1^\infty \chi_{[2m-1, 2m]}(x) \chi_{[2n-1, 2n]}(x) x^{-1-2\sigma} \, dx$$

$$= \sum_{n=1}^{2n-1} x^{-1-2\sigma} \, dx = \frac{(1 - 2^{1-2\sigma}) \zeta(2\sigma)}{2\sigma}$$

in view of (6), and (1) follows. \Box

For the Proof of Theorem 2 we need the following
Lemma. Let $0 < a < b$. If $f(x)$ is integrable on $[a, b]$, then
\begin{align*}
\left( \int_{a}^{b} f(x)x^{-s} \, dx \right)^2 &= \int_{a}^{b} \int_{a}^{b} x^{-s} \int_{a}^{b} f(u)f \left( \frac{x}{u} \right) \frac{du}{u} \, dx + \int_{a}^{b} \int_{a}^{b} x^{-s} \int_{a}^{b} f(u)f \left( \frac{u}{x} \right) \frac{du}{u} \, dx.
\end{align*}

The identity (10) remains valid if $b = \infty$, provided the integrals in question converge, in which case the second integral on the right-hand side is to be omitted.

Proof. We write the left-hand side of (10) as the double integral
\begin{align*}
\int_{a}^{b} \int_{a}^{b} (xy)^{-s} f(x)f(y) \, dx \, dy
\end{align*}
and make the change of variables $x = X/Y, y = Y$. The Jacobian of this transformation equals $1/Y$, hence the left-hand side of (10) becomes
\begin{align*}
\int_{a}^{b} \int_{a}^{b} X^{-s} \left( \int_{\min(X/a, b)}^{\max(a, X/b)} f(Y)f \left( \frac{X}{Y} \right) \frac{dY}{Y} \right) \, dX
\end{align*}
\begin{align*}
= \int_{a}^{b} \int_{a}^{b} X^{-s} \int_{a}^{b} f(Y)f \left( \frac{X}{Y} \right) \frac{dY}{Y} \, dX + \int_{a}^{b} \int_{X/b}^{b} X^{-s} \int_{a}^{b} f(Y)f \left( \frac{X}{Y} \right) \frac{dY}{Y} \, dX,
\end{align*}
as asserted.

Proof of Theorem 2. We use (8) and the Lemma to obtain that (4) certainly holds with $\varphi(x)$ given by (3), since trivially $\varphi(x) \ll x$. To see that it holds for $\sigma > 0$, we note that
\begin{align*}
\int_{1}^{x} g(u)g \left( \frac{x}{u} \right) \frac{du}{u} = \int_{1}^{\sqrt{x}} + \int_{\sqrt{x}}^{x} = 2 \int_{\sqrt{x}}^{x} g(u)g \left( \frac{x}{u} \right) \frac{du}{u},
\end{align*}
and use (11) with
\begin{align*}
g(x) = \sum_{n=1}^{\infty} \chi_{[2n-1,2n]}(x).
\end{align*}
Note then that the integrand in $\varphi(x)$ equals $1/u$ for $2m - 1 \leq u \leq 2m, 2n - 1 \leq u \leq 2n$, and otherwise it is zero. This gives the condition
\begin{align*}
4mn - 2m - 2n + 1 \leq x < 4mn, \quad \frac{1}{2} \sqrt{x} \leq n \leq \frac{1}{2}(x + 1), \quad 1 \leq m \leq \frac{1}{2}(\sqrt{x} + 1).
\end{align*}
We also have
\begin{align*}
\int_{\sqrt{x}}^{x} \chi_{[2n-1,2m]} \left( \frac{x}{u} \right) \chi_{[2n-1,2n]}(u) \frac{du}{u} \leq \int_{2n-1}^{2n} \frac{du}{u} \leq \frac{1}{2n - 1}.
\end{align*}
Therefore
\begin{align}
\varphi(x) & \ll \sum_{m \leq \sqrt{x}} \sum_{x/(4m) < n \leq (x-1+2m)/(4m-2)} \frac{1}{n} \\
& \ll \sum_{m \leq \sqrt{x}} \frac{m}{x} \left(1 + \frac{x}{m^2}\right) \ll \log x.
\end{align}

This bound shows that the integral in (4) is absolutely convergent for \(\sigma > 0\). Thus by the principle of analytic continuation this completes the proof of Theorem 2. \(\square\)

Corollary 2 follows then from (4) and (9) on setting \(\sigma = \frac{1}{2}\).

It is interesting to note that the bound in (12) is actually of the correct order of magnitude. Namely we have

**Theorem 3.** For any given \(\varepsilon > 0\) we have
\begin{equation}
\varphi(x) = \frac{1}{4} \log x + \frac{1}{4} \log \left(\frac{\pi}{2}\right) + O_\varepsilon \left(x^{\varepsilon - \frac{1}{4}}\right).
\end{equation}

**Proof.** By (8) and the inversion formula for the Mellin transform \(m[f(x)]\) (see [2, Lemma 1]) we have, for any \(c > 0\),
\begin{equation}
\varphi(x) = \frac{1}{2\pi i} \int_{\Re s = c} \frac{(1 - 2^{1-s})^2 \zeta^2(s)x^s}{s^2} \, ds.
\end{equation}

We shift the line of integration in (14) to \(c = \varepsilon - 1/4\) with \(0 < \varepsilon < 1/8\), which clearly may be assumed. Since \(\zeta(0) = -\frac{1}{2}\) and \(\zeta'(0) = -\frac{1}{2} \log(2\pi)\), the residue at the double pole \(s = 0\) is found to be
\begin{equation}
\frac{1}{4} \log x + A, \quad A = -\zeta'(0) - \log 2 = \frac{1}{2} \log \left(\frac{\pi}{2}\right).
\end{equation}

We use the functional equation (see e.g., [1, Chapter 1]) for \(\zeta(s)\), namely
\[\zeta(s) = \chi(s)\zeta(1-s), \quad \chi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s),\]
with
\[\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma + it - \frac{1}{4}} e^{it\left(\frac{1}{4}\sigma + \frac{1}{2}\right)} \left(1 + O\left(\frac{1}{t}\right)\right) \quad (t \geq 2).\]

Let \(s = \varepsilon - \frac{1}{4} + it\). Then by absolute convergence we have
\begin{align}
& \int_{T}^{2T} \frac{(1 - 2^{1-s})^2 \zeta^2(s)x^s}{s^2} \, dt \\
& = \frac{1}{4} \log x + A, \quad A = -\zeta'(0) - \log 2 = \frac{1}{2} \log \left(\frac{\pi}{2}\right).
\end{align}
where \(d(n)\) is the number of divisors of \(n\) and

\[
F(t, n) := 2t + t \log n - 2t \log(t/2\pi), \quad \frac{d^2}{dt^2} (t \log x + F(t, n)) = -\frac{2}{t}.
\]

Hence by the second derivative test (see [1, Lemma 2.2]) the above series is

\[
\ll \sum_{n=1}^{\infty} d(n)n^{\varepsilon-5/4}T^{-2\varepsilon} = \zeta^2\left(\frac{5}{4} - 2\varepsilon\right)T^{-2\varepsilon} \ll T^{-2\varepsilon}.
\]

This shows that

\[
\int_{\text{Re} s = \varepsilon - 1/4} \frac{(1 - 2^{1-s})^2\zeta^2(s)x^s}{s^2} \, ds \ll x^{\varepsilon - 1/4},
\]

hence (13) follows from (14), (15) and the residue theorem.

In concluding, note that if we write

\[
\varphi(x) = \frac{1}{4} \log x + A + \varphi_1(x),
\]

where \(A\) is given by (15) then, for \(\text{Re} s > 0\), (4) yields

\[
s^2 \left( \frac{A}{s} + \frac{1}{4s^2} + \int_1^\infty \varphi_1(x)x^{-s-1} \, dx \right) = (1 - 2^{1-s})^2\zeta^2(s),
\]

and the above integral converges absolutely, for \(\sigma > -1/4\), in view of (13). Thus by analytic continuation it follows that, for \(\sigma > -1/4\),

\[
As + \frac{1}{4} + s^2 \int_1^\infty \varphi_1(x)x^{-s-1} \, dx = (1 - 2^{1-s})^2\zeta^2(s).
\]

REFERENCES


Katedra Matematike RGF-a Universiteta u Beogradu, 11000 Beograd, Đušina 7, Serbia and Montenegro
E-mail: eivica@ubbg.etf.bg.ac.yu, aivic@rgf.bg.ac.yu