MAXIMUM MODULE VALUES OF POLYNOMIALS ON $|z| = R$ ($R > 1$)

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Let $f(z)$ and $g(z)$ be two polynomials of degrees $m \geq 1$ and $n \geq 1$ respectively on $|z| \leq R$ ($R > 1$), and $M_f = \max_{|z|=R} |f(z)|$, $M_g = \max_{|z|=R} |g(z)|$ and $M_{fg} = \max_{|z|=R} |f(z)g(z)|$. If $z = 0$ is not a root of given polynomials, it is shown that $M_{fg} \geq \delta_1 M_f M_g$, where $\delta_1 = \frac{1}{2m} \cdot \frac{1}{2n}$. On the other hand, if $z = 0$ is $k$-multiple root of $f(z)$ and $r$-multiple root of $g(z)$, then it is proved that $M_{fg} \geq \varepsilon M_f M_g$ with $\varepsilon = \frac{1}{2m-k} \cdot \frac{1}{2n-r}$. Moreover, some generalizations have been obtained for $n$ similar polynomials.

1. INTRODUCTION

Let $f$, $g : \mathbb{C} \to \mathbb{C}$ be complex-valued polynomial functions of degrees $m \geq 1$, $n \geq 1$, respectively, of a complex variable $z$, and $M_f = \max_{|z|=1} |f(z)|$, $M_g = \max_{|z|=1} |g(z)|$ and $M_{fg} = \max_{|z|=1} |f(z)g(z)|$. It is shown (see [1]) that

$$M_{fg} \geq \nu M_f M_g \text{ with } \nu = \sin^{m} \frac{\pi}{8m} \sin^{n} \frac{\pi}{8n}.$$ 

Let $f_1, f_2, \ldots, f_n : \mathbb{C} \to \mathbb{C}$ be complex-valued polynomial functions of degrees $d_1, d_2, \ldots, d_n$, respectively, of a complex variable $z$. In [2] the following inequality is obtained:

$$M_{f_1} M_{f_2} \cdots M_{f_n} \geq k M_{f_1} f_2 \cdots f_n \geq k M_{f_1} M_{f_2} \cdots M_{f_n},$$

with $k = \left( \sin \left( \frac{2}{n} \frac{\pi}{8d_1} \right) \right)^{d_1} \cdots \left( \sin \left( \frac{2}{n} \frac{\pi}{8d_2} \right) \right)^{d_2} \cdots \left( \sin \left( \frac{2}{n} \frac{\pi}{8d_n} \right) \right)^{d_n}$.

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It shown in [3] that \( M_{fg} > \nu M_f M_g \) with \( \nu = \frac{1}{2^m} \cdot \frac{1}{2^n} \).

If \( f(z) \) and \( g(z) \) accept \( z = 0 \) as \( k \)-multiple and \( r \)-multiple roots, respectively, then in [4] the following inequality is obtained:
\[
M_{fg} \geq \delta M_f M_g \quad \text{with} \quad \delta = \frac{1}{2^{m-k}} \frac{1}{2^{n-r}}.
\]

In [5], some generalizations of the results of [3] and [4], have been obtained for \( n \) similar polynomials.

2. MAXIMUM MODULE VALUES OF POLYNOMIALS NOT ADMITTING \( z = 0 \) AS A ROOT ON \( |z| \leq R \) (\( R > 1 \))

Firstly, we give two lemmas in order to facilitate the development of our work.

**Lemma 2.1.** For \( |z| = R \) and \( |\gamma| \neq 1/R \), we have
\[
\left| \frac{R^2 \gamma - z}{1 - \gamma z} \right| = R.
\]

For the proof, it is enough to take the module of the both sides of
\[
\frac{R^2 \gamma - z}{1 - \gamma z} = \frac{R^2 \gamma - z}{R^2 (z - R^2 \gamma)}.
\]

**Lemma 2.2.** We have
\begin{enumerate}
  \item \((z - R^2 \gamma) = \left(z - \frac{1}{\gamma}\right) \frac{R^2 \gamma - z}{1 - \gamma z} \text{ for } |\gamma| \neq \frac{1}{R}.
  \item All roots of the polynomials \( f(z) = (z - R^2 \alpha_1)(z - R^2 \alpha_2) \cdots (z - R^2 \alpha_k) \) of degree \( m \geq 1 \) satisfy \( |z| \leq R, \) where \( |\alpha_k| \leq 1/R \), \( k = 1, 2, \ldots, m \).
\end{enumerate}

**Proof.** (i) is obvious and (ii) follows from the hypothesis and \( |R^2 \alpha_k| = R^2 |\alpha_k|, \) \( k = 1, 2, \ldots, m \).

**Theorem 2.1.** Let \( M_f = \max_{|z|=R} |f(z)|, \) \( M_g = \max_{|z|=R} |g(z)|, \) \( M_{fg} = \max_{|z|=R} |f(z)g(z)| \) be the maximum module values of the polynomials
\[
f(z) = \prod_{i=1}^{m} (z - R^2 \alpha_i) \quad (\alpha_i \neq 0, \ |\alpha_i| \leq 1/R)
\]
and
\[
g(z) = \prod_{j=1}^{n} (z - R^2 \beta_j) \quad (\beta_j \neq 0, \ |\beta_j| \leq 1/R)
\]
on \( |z| = R \). Then
\[
M_{fg} \leq \delta_1 M_f M_g, \quad \text{where} \quad \delta_1 = \frac{1}{2^m} \frac{1}{2^n}.
\]
Proof. Consider the polynomial

\[ h(z) = \prod_{k=1}^{\ell} (z - z_k) \quad (z_k \neq 0, \ |z_k| \leq R). \]

Then we have \( \mathcal{M}_h = R^\ell \max_{|z|=R} \left| \frac{h(z)}{z^\ell} \right| = R^\ell \max_{|z|=R} \left| \prod_{k=1}^{\ell} \left(1 - \frac{z_k}{z}\right) \right| \). If we put

\[ t = R/z, \] 

then we have \( s(t) = \prod_{k=1}^{\ell} \left(1 - \frac{z_k t}{R}\right) \) it comes \( \mathcal{M}_h = R^\ell \max_{|t|\leq 1} |s(t)| \), where \( s(0) = 1 \), and we obtain from the Maximum module principle \( \mathcal{M}_h \geq R^\ell \). Furthermore, by the definition of \( \mathcal{M}_h \) it is clear that \( \mathcal{M}_h \leq 2^\ell R^\ell \).

Since \( f(z) \) and \( g(z) \) are polynomials of the type (2) similar argument yields \( \mathcal{M}_f \leq 2^m R^m, \quad \mathcal{M}_g \leq 2^n R^n \).

If \( z_1 = z_2 = \cdots = z_t = Re^{i\theta_0} (\theta_0 \in \mathbb{R}) \), then \( \mathcal{M}_h = 2^\ell R^\ell \). On the other hand, let us consider the following sequences:

\[ R^2\alpha_1, \ldots, R^2\alpha_{p-1}/\alpha_p, \ldots, \alpha_m; \quad |\alpha_p| > 1/R, \ldots, |\alpha_m| > 1/R, \]
\[ R^2\beta_1, \ldots, R^2\beta_{q-1}/\beta_q, \ldots, \beta_n; \quad |\beta_q| > 1/R, \ldots, |\beta_n| > 1/R. \]

Let

\[ F(z) = \prod_{\mu=p}^{p-1} (z - R^2\alpha_\mu) \frac{m}{j=1} \left(z - \frac{1}{\alpha_j}\right), \quad G(z) = \prod_{i=1}^{q-1} (z - R^2\beta_i) \frac{n}{j=q} \left(z - \frac{1}{\beta_j}\right) \]

be polynomials on \( |z| \leq R (R > 1) \) with \( m, n \geq 1 \). Then, if we write \( \mathcal{A} = \alpha_p \cdots \alpha_m, \quad \mathcal{B} = \beta_q \cdots \beta_n \), we have by means of Lemma 2.1 and Lemma 2.2

\[ f(z) = \mathcal{A} F(z) \frac{m}{\mu=p} \left(\frac{R^2\alpha_\mu - z}{1 - \alpha_\mu z}\right), \quad g(z) = \mathcal{B} G(z) \frac{n}{q=q} \left(\frac{R^2\beta_q - z}{1 - \beta_q z}\right). \]

It is easily deduced from the last equalities that we have

\[ \mathcal{M}_f = |\mathcal{A}| \mathcal{M}_F R^{m-p}, \quad \mathcal{M}_g = |\mathcal{B}| \mathcal{M}_G R^{n-q}, \quad \mathcal{M}_{fg} = |\mathcal{A}| |\mathcal{B}| \mathcal{M}_{FG} R^{m-p+n-q} \]

and hence

\[ \frac{\mathcal{M}_{fg}}{\mathcal{M}_f \mathcal{M}_g} = \frac{\mathcal{M}_{FG}}{\mathcal{M}_F \mathcal{M}_G}. \]

Since \( F(z) \) and \( G(z) \) are polynomials of type (2), we obtain \( \mathcal{M}_F \leq 2^m R^m, \quad \mathcal{M}_G \leq 2^n R^n \) and \( \mathcal{M}_{FG} \geq R^{m+n} \), and thus (1) is found from (3).

Corollary 2.1. Let \( f_1(z), f_2(z), \ldots, f_n(z) \) be polynomials of degrees \( m_1, m_2, \ldots, m_n \), respectively, on \( |z| \leq R (R > 1) \). Suppose that \( z = 0 \) is not a root of these polynomials. Then

\[ \mathcal{M}_{f_1 f_2 \cdots f_n} \geq \varepsilon \mathcal{M}_{f_1} \mathcal{M}_{f_2} \cdots \mathcal{M}_{f_n}, \quad \text{where} \quad \varepsilon = \frac{1}{2^{m_1}} \frac{1}{2^{m_2}} \cdots \frac{1}{2^{m_n}}. \]
3. MAXIMUM MODULE VALUES OF POLYNOMIALS HAVING \( z = 0 \) AS A ROOT ON \( |z| \leq R \) (\( R > 1 \))

In this section, we will give some relations concerning maximum module values of polynomials which admit \( z = 0 \) as a simple or multiple root on \( |z| = R \) (\( R > 1 \)).

**Theorem 3.1.** Let

\[
f(z) = z \prod_{i=1}^{m-1} (z - R^{2} \alpha_{i}) \quad (\alpha_{i} \neq 0, \ |\alpha_{i}| \leq 1/R)
\]

and

\[
g(z) = z \prod_{j=1}^{n} (z - R^{2} \beta_{j}) \quad (\beta_{j} \neq 0, \ |\beta_{j}| \leq 1/R)
\]

be polynomials on \( |z| \leq R \) (\( R > 1 \)) with \( m - 1, n - 1 \geq 1 \). Then

\[
\mathcal{M}_{fg} \geq \delta_{2} \mathcal{M}_{f} \mathcal{M}_{g}, \quad \text{where} \quad \delta_{2} = \frac{1}{2^{m-1}} \frac{1}{2^{n-1}}.
\]

**Proof.** Consider

\[
h(z) = z \prod_{k=1}^{\ell-1} (z - z_{k}) \quad (z_{k} \neq 0, \ |z_{k}| \leq R).
\]

If we apply the technique used in Theorem 2.1, then we have \( \mathcal{M}_{h} \geq R^{\ell} \) and \( \mathcal{M}_{h} \leq 2^{\ell-1} R^{\ell} \). Similarly, we can find \( \mathcal{M}_{f} \leq 2^{m-1} R^{m} \), \( \mathcal{M}_{g} \leq 2^{n-1} R^{n} \).

If \( z_{1} = z_{2} = \cdots = z_{\ell-1} = Re^{i\theta_{0}} \ (\theta_{0} \in \mathbb{R}) \), then \( \mathcal{M}_{h} = 2^{\ell-1} R^{\ell} \). Now, let us write the following sequences:

\[
0, R^{2}\alpha_{1}, \ldots, R^{2}\alpha_{p-1}/\alpha_{p}, \ldots, \alpha_{m-1}; \quad |\alpha_{p}| > 1/R, \ldots, |\alpha_{m-1}| > 1/R,
\]

\[
0, R^{2}\beta_{1}, \ldots, R^{2}\beta_{q-1}/\beta_{q}, \ldots, \beta_{n-1}; \quad |\beta_{q}| > 1/R, \ldots, |\beta_{n-1}| > 1/R.
\]

As in Theorem 2.1, consider

\[
F_{i}(z) = z \prod_{i=1}^{p-1} (z - R^{2} \alpha_{i}) \prod_{j=p}^{m-1} \left( z - \frac{1}{\alpha_{j}} \right), \quad G_{i}(z) = z \prod_{i=1}^{q-1} (z - R^{2} \beta_{i}) \prod_{j=q}^{n-1} \left( z - \frac{1}{\beta_{j}} \right).
\]

Putting \( A_{1} = \prod_{i=p}^{m} \alpha_{i}, \ B_{1} = \prod_{i=q}^{n} \beta_{i} \), we can write

\[
f(z) = A_{1} F_{i}(z) \prod_{i=p}^{m-1} \left( R^{2} \alpha_{i} - z \right) \left( 1 - \frac{1}{\alpha_{i} z} \right), \quad g(z) = B_{1} G_{i}(z) \prod_{i=q}^{n-1} \left( R^{2} \beta_{i} - z \right) \left( 1 - \frac{1}{\beta_{i} z} \right),
\]

and hence

\[
\mathcal{M}_{f} = |A_{1}| \mathcal{M}_{F_{i}} R^{m-1-p}, \quad \mathcal{M}_{g} = |B_{1}| \mathcal{M}_{G_{i}} R^{n-1-q},
\]

\[
\mathcal{M}_{fg} = |A_{1}| |B_{1}| \mathcal{M}_{F_{i}G_{i}} R^{m+n-p-q-2}.
\]
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It is clear that the following equation results from the last equalities:

$$
\frac{M_{fg}}{M_f M_g} = \frac{M_{F_2 G_1}}{M_{F_1} M_{G_1}}.
$$

(7)

Since $F_1(z)$ and $G_1(z)$ are polynomials of the type (6), we have $M_{F_1} \leq 2^{m-1} R^n$, $M_{G_1} \leq 2^{n-1} R^n$ and $M_{F_1 G_1} \geq R^{m+n}$. Thus (5) is obtained from (7).

**Corollary 3.1.** Let $f_1(z), f_2(z), \ldots, f_n(z)$ be polynomials of degrees $m_1, m_2, \ldots, m_n$, respectively, on $|z| \leq R$ ($R > 1$). Suppose that $z = 0$ is not simple zero of these polynomials.

Then

$$
M_{f_1 f_2 \cdots f_n} \geq \epsilon_1 M_{f_1} M_{f_2} \cdots M_{f_n}, \text{ where } \epsilon_1 = \frac{1}{2^{m_1-1}} \frac{1}{2^{m_2-1}} \cdots \frac{1}{2^{m_n-1}}.
$$

(8)

**Theorem 3.2.** Let

$$
f(z) = z^k \prod_{i=1}^{m-k} \left(z - R^2 \alpha_i\right) \quad (\alpha_i \neq 0, |\alpha_i| \leq 1/R)
$$

and

$$
g(z) = z^r \prod_{j=1}^{n-r} \left(z - R^2 \beta_j\right) \quad (\beta_j \neq 0, |\beta_j| \leq 1/R)
$$

be polynomials on $|z| \leq R$ ($R > 1$). Then

$$
M_{fg} \geq \delta M_f M_g, \text{ for } \delta = \frac{1}{2^{m-k}} \frac{1}{2^{n-r}}.
$$

(9)

**Proof.** Consider $h(z) = z^w \prod_{k=1}^{w-s} (z - z_k)$ on $|z| \leq R$. The following inequalities are easily found:

$$
M_h \geq R^w, \quad M_h \leq 2^{w-s} R^w \text{ and } M_f \leq 2^{m-k} R^k, \quad M_g \leq 2^{n-r} R^n.
$$

Let us form now the following polynomials on the circle $|z| \leq R$:

$$
F_2(z) = z^k \prod_{i=1}^{m-k} (z - R^2 \alpha_i) \prod_{j=p}^{m-k} \left(z - \frac{1}{\alpha_j}\right), \quad G_2(z) = z^r \prod_{j=q}^{n-r} (z - R^2 \beta_j) \prod_{j=q}^{n-r} \left(z - \frac{1}{\beta_j}\right).
$$

Taking $A_2 = \alpha_p \cdots \alpha_{m-k}, \quad B_2 = \beta_q \cdots \beta_{n-r}$, we can write

$$
f(z) = A_2 F_2(z) \prod_{\mu=p}^{m-k} \left(R^2 \alpha_\mu - z \right) \left(1 - \alpha_\mu z\right), \quad g(z) = B_2 G_2(z) \prod_{\eta=q}^{n-r} \left(R^2 \beta_\eta - z \right) \left(1 - \beta_\eta z\right).
$$

From these equalities the following is deduced:

$$
M_f = |A_2| M_{F_2} R^{m-k-p}, \quad M_g = |B_2| M_{G_2} R^{n-r-q},
$$

$$
M_{fg} = |A_2||B_2| M_{F_2 G_2} R^{m+n-k-p-r-q}.
$$
and

\[ \frac{M_{fg}}{M_f M_g} = \frac{M_{F_2G_2}}{M_{F_2}M_{G_2}}. \]

But, on the other hand we have \( M_{F_2} \leq 2^{m-k}R^m, M_{G_2} \leq 2^{n-r}R^n \) and \( M_{F_2G_2} \geq R^{m+n} \). Thus (9) is obtained from (10).

**Corollary 3.2.** Let \( f_1(z), f_2(z), \ldots, f_n(z) \) be polynomials of degrees \( m_1, m_2, \ldots, m_n \), and suppose that each one accepts \( z = 0 \) as \( r_i \) \((i = 1, 2, \ldots, n)\) multiplicity root, respectively. When \( \varepsilon_2 = \frac{1}{2^{m_1-r_1}} \frac{1}{2^{m_2-r_2}} \cdots \frac{1}{2^{m_n-r_n}} \), then

\[ M_{f_1f_2\cdots f_n} \geq \varepsilon_2 M_{f_1}M_{f_2} \cdots M_{f_n}. \]

**Result.** For \( \varepsilon_2 = 1 \) it is necessary and sufficient that

\[ f_1(z) = z^{m_1}, f_2(z) = z^{m_2}, \ldots, f_n(z) = z^{m_n}. \]

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