AN INEQUALITY FOR THE LEBESGUE MEASURE

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In this paper we present a new inequality for the Lebesgue measure and give some of its applications.

If \( \lambda \) is the Lebesgue measure on the set of real numbers \( \mathbb{R} \) and \( \{A_n\} \) sequences of Lebesgue measurable sets in \( \mathbb{R} \), then we have the following inequality:

\[
\lambda (\lim A_n) \leq \lim \lambda (A_n).
\]

But for the inequality \( \lim \lambda (A_n) \leq \lambda (\lim A_n) \) we must suppose that \( \lambda (\bigcup_{n=1}^{\infty} A_n) < \infty \) for at least one value of \( n \) (see [1], p 40.). Example: for a family of intervals \( I_n = [n, n+1) \) \( n = 0, 1, \ldots \) we have: \( \lim \lambda (A_n) = 1 \) and \( \lambda (\lim A_n) = 0 \).

But in the general case we have the following inequality:

**Proposition.** Let \( A \) be a measurable set of a positive measure and \( \{x_n\} \) a bounded sequence of real numbers. Then

\[
(1) \quad \lambda(A) \leq \lambda (\lim (x_n + A)).
\]

**Proof.** Let \( K \subseteq A \) be a compact set. From \( \lim (x_n + K) \subseteq \lim (x_n + A) \) follows \( \lambda (\lim (x_n + K)) \leq \lambda (\lim (x_n + A)) \). Now we have

\[
(2) \quad \lambda(K) = \lim \lambda ((x_n + K)) \leq \lambda (\lim (x_n + K)) \leq \lambda (\lim (x_n + A))
\]

Since \( \lambda(A) = \sup \{\lambda(K) : K \text{ is the compact subset of } A\} \), (2) implies (1).

**Corollary 1.** (H. Steinhaus [2]) Let \( A \) be a Lebesgue measurable set of a positive measure. Then in \( A \) exist at least two points such that distance between them is a rational number.

**Proof.** Let \( (q_n) \) be arbitrary bounded sequence of rational numbers, such that their member are different. From

\[
0 < \lambda(A) \leq \lambda (\lim (A + q_n))
\]

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it follows that the set \( \lim (A + q_n) \) is nonempty. So, there exists \( p_1, p_2 \in A \) and \( i, j \) such that \( p_1 + q_i = p_2 + q_j \), which implies \( |p_1 - p_2| = |q_i - q_j| \).

As a further practical application of the inequality (1) we give the following short and simple proof of the famous Steinhaus’ result.

**Corollary 2.** (H. Steinhaus [2]) Let \( A \subseteq \mathbb{R} \) be a Lebesgue measurable set of a positive measure. Then its difference set \( A - A = \{ x \mid x = a_1 - a_2, \ a_1, a_2 \in A \} \) contains a neighborhood of zero.

**Proof.** Assume that the statement is wrong. Then there exists compact set of positive measure \( K \subseteq A \) such that the difference set \( K - K \) does not contain a neighborhood of zero. It follows that there exists convergent sequence \( \{x_n\} \subseteq \mathbb{R} \) such that \( \lim x_n = 0 \) and \( \{x_n\} \cap (K - K) = \emptyset \). From

\[
0 < \lambda(K) \leq \lambda(\lim (K - x_n))
\]

it follows that the set \( \lim (K - x_n) \) is nonempty which implies that there exists \( t \in \mathbb{R} \) such that \( \{x_n + t\} \subseteq K \) for infinitely many values of \( n \). From \( \lim x_n = 0 \) it follows that \( t \in K \), as \( K \) is a closed set. Thus we have that there exists infinite sequences \( \{a_j\} \subseteq K \) and \( \{x_{n_j}\} \subseteq \{x_n\} \) such that \( x_{n_j} = a_j - t \in K - K \), which is a contradiction.

**REFERENCES**