TWO MAPPINGS RELATED TO HÖLDER’S INEQUALITY

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In this paper, by the Hölder’s inequalities we define two mappings, investigate its monotonicity and gave some refinements for Hölder’s inequalities.

1. INTRODUCTION AND MAIN RESULTS

Let \( a_i > 0, b_i > 0 \) (\( i = 1, 2, \ldots, n \)), \( p \) and \( q \) be two non-zero real numbers such that \( p^{-1} + q^{-1} = 1 \). If \( p > 1 \), then

\[
\sum_{i=1}^{n} a_ib_i \leq \left( \sum_{i=1}^{n} a_i^p \right)^{1/p} \left( \sum_{i=1}^{n} b_i^q \right)^{1/q},
\]

if \( p < 1 \) (\( p \neq 0 \)), then the inequality in (1.1) is reversed.

The inequality (1.1) is called the Hölder’s inequality (see [6–8]).

Let \( f, g : [a, b] \to \mathbb{R} \) (\( a < b \)) be integrable and positive functions, \( p \) and \( q \) be two non-zero real numbers such that \( p^{-1} + q^{-1} = 1 \). If \( p > 1 \), then

\[
\int_{a}^{b} f(s) g(s) \, ds \leq \left( \int_{a}^{b} f^p(s) \, ds \right)^{1/p} \left( \int_{a}^{b} g^q(s) \, ds \right)^{1/q},
\]

if \( p < 1 \) (\( p \neq 0 \)), then the inequality in (1.2) is reversed.

The inequality (1.2) is integral form of inequality (1.1) (see [6–8]).

We define two mappings \( H \) and \( h \) by

\[
H : \mathbb{N}_+ \to \mathbb{R}, \quad H(n) = \left( \sum_{i=1}^{n} a_i^p \right)^{1/p} \left( \sum_{i=1}^{n} b_i^q \right)^{1/q} - \sum_{i=1}^{n} a_i b_i
\]

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and
\[ h : [a, b] \times [a, b] \to \mathbb{R}, \quad h(x, y) = \left( \int_a^y f^p(s) \, ds \right)^{1/p} \left( \int_a^y g^q(s) \, ds \right)^{1/q} + \int_a^y f(s) g(s) \, ds. \] 

The aim of this paper is to study monotonicity properties of \( H \) and \( h \), and the obtain some refinements of (1.1) and (1.2) by the these properties. The results from other inequalities connected with (1.1) and (1.2) can be seen in [8, p. 3-28].

The monotonicity property of the mapping \( H \) are embodied in the following theorem.

**Theorem 1.1.** Let \( a_i > 0, b_i > 0 \) \( (i = 1, 2, \ldots, n; n > 1) \), \( p \) and \( q \) be two non-zero real numbers such that \( p^{-1} + q^{-1} = 1 \). We define \( C \) by
\[ C(k) = \frac{1}{k} \left( \sum_{i=1}^k a_i^p \right)^{1/p} \left( \sum_{i=1}^k b_i^q \right)^{1/q} + \sum_{i=k+1}^n a_i b_i, \quad k = 1, 2, \ldots, n; \quad \sum_{i=n+1}^a a_i b_i = 0. \]

When \( p > 1 \), we have (1):
\[ H(n) \geq H(n-1), \]
and (2):
\[ \sum_{i=1}^n a_i b_i = C(1) \leq C(2) \leq \cdots \leq C(n-1) \leq C(n) = \left( \sum_{i=1}^n a_i^p \right)^{1/p} \left( \sum_{i=1}^n b_i^q \right)^{1/q}. \]

The inequalities in (1.3) and (1.4) are reversed for \( p < 1(p \neq 0) \).

The monotonicity properties of the mapping \( h \) are given in the following theorem.

**Theorem 1.2.** Let \( f, g : [a, b] \to \mathbb{R} \) be integrable and positive functions, \( p \) and \( q \) be two non-zero real numbers such that \( p^{-1} + q^{-1} = 1 \). Then we obtain the following.

1. When \( p > 1 \), mapping \( h(x, b) \) is monotonically decreasing, \( h(x, b) \) is monotonically increasing for \( p < 1(p \neq 0) \), on \([a, b] \) with \( x \);
2. When \( p > 1 \), mapping \( h(a, y) \) is monotonically increasing, \( h(a, y) \) is monotonically decreasing for \( p < 1(p \neq 0) \), on \([a, b] \) with \( y \);
3. For \( \forall x \in (a, b) \), when \( p > 1 \), we have
\[
\begin{align*}
\int_a^b f(s) g(s) \, ds & \leq \left( \int_a^x f^p(s) \, ds \right)^{1/p} \left( \int_a^x g^q(s) \, ds \right)^{1/q} + \int_a^x f(s) g(s) \, ds \\
& \leq \left( \int_a^b f^p(s) \, ds \right)^{1/p} \left( \int_a^b g^q(s) \, ds \right)^{1/q}, \\
\int_a^b f(s) g(s) \, ds & \leq \left( \int_a^b f^p(s) \, ds \right)^{1/p} \left( \int_a^b g^q(s) \, ds \right)^{1/q} + \int_a^x f(s) g(s) \, ds \\
& \leq \left( \int_a^b f^p(s) \, ds \right)^{1/p} \left( \int_a^b g^q(s) \, ds \right)^{1/q}.
\end{align*}
\]
Two mappings related to H"older’s inequalities

\( \int_a^b f(s)g(s) \, ds \leq \frac{1}{2} \left( \left( \int_a^b f^p(s) \, ds \right)^{1/p} \left( \int_a^b g^q(s) \, ds \right)^{1/q} + \int_a^b f(s)g(s) \, ds \right) \)

\[ (1.7) \]

\[ \leq \left( \int_a^b f^p(s) \, ds \right)^{1/p} \left( \int_a^b g^q(s) \, ds \right)^{1/q}. \]

The inequalities in (1.5), (1.6) and (1.7) are reversed for \( p < 1 \) (\( p \neq 0 \)).

Remark. (1.4) and (1.5)–(1.7) are refinements of (1.1) and (1.2), respectively.

2. PROOF OF THEOREMS

Proof of Theorem 1.1. (1) From \( p^{-1} + q^{-1} = 1 \), we have

\[ \left( \sum_{i=1}^{m} a_i^p \right)^{1/p} \left( \sum_{i=1}^{m} b_i^q \right)^{1/q} \]

\[ = \left( \sum_{i=1}^{m} b_i^q \right) \left( \left( \sum_{i=1}^{m} b_i^q \right)^{-1} \sum_{i=1}^{m} b_i^q \left( a_i b_i^{1/(p-1)} \right)^p \right)^{1/p} \]

\[ (2.1) \]

\( (m = n - 1, n). \)

When \( p > 1 \) i.e. \( 0 < p^{-1} < 1 \), \( x^{1/p} \) is concave function on \( (0, +\infty) \) with \( x \).

Using JENSEN’s inequality of concave function (it is reversed of JENSEN’s inequality of convex function, see [1–3]), we get

\[ \left( \sum_{i=1}^{n} b_i^q \right) \left( \left( \sum_{i=1}^{n} b_i^q \right)^{-1} \sum_{i=1}^{n} b_i^q \left( a_i b_i^{1/(p-1)} \right)^p \right)^{1/p} \]

\[ = \left( \sum_{i=1}^{n} b_i^q \right) \left( \left( \sum_{i=1}^{n} b_i^q \right)^{-1} \left( \sum_{i=1}^{n-1} b_i^q \right) \left( \sum_{i=1}^{n-1} b_i^q \right)^{-1} \sum_{i=1}^{n-1} b_i^q \left( a_i b_i^{1/(p-1)} \right)^p \right)^{1/p} \]

\[ + b_n^q \left( a_n b_n^{1/(p-1)} \right)^p \]

\[ (2.2) \]

\[ \geq \left( \sum_{i=1}^{n-1} b_i^q \right) \left( \left( \sum_{i=1}^{n-1} b_i^q \right)^{-1} \sum_{i=1}^{n-1} b_i^q \left( a_i b_i^{1/(p-1)} \right)^p \right)^{1/p} + b_n^q \left( a_n b_n^{1/(p-1)} \right)^p. \]
Using (2.1) and (2.2), we obtain

\[
H(n) - H(n - 1)
= \left( \sum_{i=1}^{n} a_i^{p} \right)^{1/p} \left( \sum_{i=1}^{n} b_i^{q} \right)^{1/q} - \sum_{i=1}^{n} a_i b_i - \left( \sum_{i=1}^{n-1} a_i^{p} \right)^{1/p} \left( \sum_{i=1}^{n} b_i^{q} \right)^{1/q} \\
= \left( \sum_{i=1}^{n} a_i^{p} \right)^{1/p} \left( \sum_{i=1}^{n-1} b_i^{q} \right)^{1/q} + \sum_{i=1}^{n-1} a_i b_i = \left( \sum_{i=1}^{n} b_i^{q} \right)^{1/q} \left( \sum_{i=1}^{n} a_i^{p} \right)^{1/p} - \left( \sum_{i=1}^{n-1} b_i^{q} \right)^{1/q} a_n b_n \\
= \left( \sum_{i=1}^{n} a_i^{p} \right)^{1/p} \left( \sum_{i=1}^{n-1} b_i^{q} \right)^{1/q} + a_n b_n - \left( \sum_{i=1}^{n-1} a_i^{p} \right)^{1/p} \left( \sum_{i=1}^{n} b_i^{q} \right)^{1/q} - a_n b_n \\
= 0,
\]

which implies the inequality (1.3).

When \(0 < p < 1\) i.e. \(p^{-1} > 1\), \(x^{1/p}\) is convex function on \((0, +\infty)\) with \(x\). By the JENSEN’s inequality of convex function (see [1–3]), we get that the inequality in (2.2) is reversed. Then the inequality in (2.3) is reversed, that implies the inequality in (1.3) is reversed.

When \(p < 0\), by \(p^{-1} + q^{-1} = 1\) we have \(0 < q < 1\). Using case of \(0 < p < 1\) and symmetry of \(p\) and \(q\), we get that the inequality in (1.3) is also reversed.

(2) When \(p > 1\), from (1.3), we have

\[
H(1) \leq H(2) \leq \cdots \leq H(k) \leq \cdots \leq H(n - 1) \leq H(n).
\]

By \(C(k) = H(k) + \sum_{i=1}^{n} a_i b_i \) \((k = 1, 2, \ldots, n)\), expression (2.4) plus \(\sum_{i=1}^{n} a_i b_i\) yields (1.4).

When \(p < 1\) \((p \neq 0)\), the inequalities in (2.4) are reversed, that implies the inequalities in (1.4) are reversed.

The proof of Theorem (1.1) is completed.

**Proof of Theorem 1.2.**

(1) Case 1. \(p > 1\). For any \(x_1, x_2 \in [a, b], x_1 < x_2\), when \(x_2 < b\), from \(p^{-1} + q^{-1} = 1\), we have for \(i = 1, 2\)

\[
\left( \int_{x_i}^{b} f^p(s) \, ds \right)^{1/p} \left( \int_{x_i}^{b} g^q(s) \, ds \right)^{1/q} = \left( \int_{x_i}^{b} g^q(s) \, ds \right)^{1/q} \left( \int_{x_i}^{b} f^p(s) \, ds \right)^{1/p}.
\]
By the concavity of \( x^{1/p} \) and JENSEN’S inequality of concave function, we get

\[
\left( \int_{x_1}^{b} g^q(s) \, ds \right) \left( \frac{\int_{x_1}^{b} f^p(s) \, ds}{\int_{x_1}^{b} g^q(s) \, ds} \right)^{1/p} = \left( \int_{x_1}^{b} g^q(s) \, ds \right) \left( \frac{\int_{x_1}^{b} f^p(s) \, ds}{\int_{x_1}^{b} g^q(s) \, ds} \right)^{1/p}
\]

(2.6)

\[
\geq \left( \int_{x_2}^{b} g^q(s) \, ds \right) \left( \frac{\int_{x_2}^{b} f^p(s) \, ds}{\int_{x_2}^{b} g^q(s) \, ds} \right)^{1/p} = \left( \int_{x_2}^{b} g^q(s) \, ds \right) \left( \frac{\int_{x_2}^{b} f^p(s) \, ds}{\int_{x_2}^{b} g^q(s) \, ds} \right)^{1/p}.
\]

By JENSEN’S integral inequality of concave function (it is reversed of JENSEN’S integral inequality of convex function, see [5, 6]), we have

\[
\left( \frac{\int_{x_1}^{b} f^p(s) \, ds}{\int_{x_1}^{b} g^q(s) \, ds} \right)^{1/p} = \left( \frac{1}{\int_{x_1}^{b} g^q(s) \, ds} \int_{x_1}^{b} g^q(s) \left( f(s) \left( g(s) \right)^{-1/(p-1)} \right)^p \, ds \right)^{1/p}
\]

(2.7)

\[
\geq \frac{1}{\int_{x_2}^{b} g^q(s) \, ds} \int_{x_1}^{b} g^q(s) \left( f(s) \left( g(s) \right)^{-1/(p-1)} \right)^p \, ds
\]

Using (2.5), (2.6) and (2.7), we obtain

\[
h(x_1, b) = \left( \int_{x_1}^{b} f^p(s) \, ds \right)^{1/p} \left( \int_{x_1}^{b} g^q(s) \, ds \right)^{1/q} - \int_{x_1}^{b} f(s)g(s) \, ds
\]

(2.8)

\[
\geq \left( \int_{x_2}^{b} g^q(s) \, ds \right) \left( \frac{\int_{x_2}^{b} f^p(s) \, ds}{\int_{x_2}^{b} g^q(s) \, ds} \right)^{1/p} - \int_{x_2}^{b} f(s)g(s) \, ds
\]

\[
= \left( \int_{x_2}^{b} f^p(s) \, ds \right)^{1/p} \left( \int_{x_2}^{b} g^q(s) \, ds \right)^{1/q} - \int_{x_2}^{b} f(s)g(s) \, ds
\]

\[
= h(x_2, b).
\]

When \( x_2 = b \), by inequality (1.2) we have

\[
h(x_1, b) \geq 0 = h(b, b) = h(x_2, b).
\]

The (2.8) and (2.9) imply that \( h(x, b) \) is monotonically decreasing on \([a, b]\) with \( x \).

Case 2. \( p < 1(p \neq 0) \). When \( 0 < p < 1 \), by the convexity of \( x^{1/p} \), we get that the inequalities in (2.6), (2.7) and (1.2) are reversed. Then the inequalities in (2.8) and (2.9) are reversed, that imply \( h(x, b) \) is monotonically increasing on \([a, b]\) with \( x \). When \( p < 0 \), we have \( 0 < q < 1 \) by \( p^{-1} + q^{-1} = 1 \). Using case of \( 0 < p < 1 \) and...
symmetry of \( p \) and \( q \), we get that \( h(x, b) \) is also monotonically increasing on \([a, b]\) with \( x \).

(2) By the same arguments of proof for case (1) in Theorem 1.2, we can prove monotonicity property of \( h(a, y) \) with \( y \).

(3) Let any \( x \in (a, b) \). When \( p > 1 \), by monotonically increasing of \( h(a, y) \) on \([a, b]\) with \( y \), we have

\[
(2.10) \quad 0 = h(a, a) \leq h(a, x) \leq h(a, b),
\]

expression \((2.10)\) plus \( \int_a^b f(s)g(s)\,ds \) yields \((1.5)\). By monotonically decreasing of \( h(x, b) \) on \([a, b]\) with \( x \), we have

\[
(2.11) \quad 0 = h(b, b) \leq h(x, b) \leq h(a, b),
\]

expression \((2.11)\) plus \( \int_a^b f(s)g(s)\,ds \) yields \((1.6)\); Expression \((1.5)\) plus \((1.6)\) yields \((1.7)\).

When \( p < 1(p \neq 0) \), the inequalities in \((2.10)\) and \((2.11)\) are reversed, that imply the inequalities in \((1.5), (1.6)\) and \((1.7)\) are reversed.

The proof of Theorem 1.2 is completed.

REFERENCES


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