ESTIMATES ON THE SIZE OF THE DOMAIN OF WEAK INVERTIBILITY IN A FORM OF THE INVERSE MAPPING THEOREM

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Let $f : E \rightarrow E$ be a mapping of a normed space $E$ to itself. Let the derivative $D$ at $a \in E$ be a bounded isomorphism on $E$ with bounded inverse $D^{-1}$. Assume, in addition, that there exist constants $C, r, \alpha > 0$ such that

$$\|f(a + h) - f(a) - Dh\| \leq C\|h\|^{1+\alpha}$$

for $\|h\| \leq r$.

We show that $f(a + h) \neq f(a)$ whenever

$$0 < \|h\| < \min\left(r, \left(\frac{1}{C}\|D^{-1}\|^{-1}\right)^{1/\alpha}\right).$$

In addition certain stability results of the nonlinear mapping $f$ are established.

1. INTRODUCTION

Let $f : E \rightarrow E$ be a mapping from the normed space $E$ to itself. Recall that the (Fréchet) derivative, $D = D_a$, of $f$ at $a \in E$ is a bounded linear map from $E$ to $E$ such that

$$\|f(a + h) - f(a) - Dh\| = o(\|h\|).$$

That $D$ is a bounded isomorphism on $E$ with bounded inverse $D^{-1}$ is not enough to ensure local diffeomorphism of $f$ near $a$. What is needed, in addition, is the continuity of $x \rightarrow D_x$ near $a$ and the completeness of $E$.

2000 Mathematics Subject Classification: 47J07
Keywords and Phrases:

†Supported by NSF grant DMS 9977116.
It is of interest, however, to note that if these assumptions are not fulfilled, we still have a weak form of the inverse mapping theorem:

**Theorem 1.1.** Let $f : E \to E$ be a mapping from the normed space $E$ to itself. Assume that the derivative $D$ at $a \in E$ is a bounded isomorphism on $E$ with bounded inverse $D^{-1}$. Then there exists $\epsilon > 0$ such that

\[ 0 < \|h\| < \epsilon \implies f(a + h) \neq f(a). \]

It appears that a result of this form was proved first by Dieudonné [2] in the slightly more restrictive setting where $E$ is a Banach space. Moreover he also assumed that the mapping $f$ is continuous in a neighborhood of the point $a$. In the above theorem we do not make this assumption. See also [1] where this result is proved in the finite dimensional case. In a slightly different context Leach [3] has proved results of a similar nature.

As a simple example of the application of Theorem 1.1, consider

\[ f(x) = \begin{cases} x + 2x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \]

We have $f'(0) = 1$ and the lack of continuity of $f'(x)$ near 0 explains the noninvertibility of the restriction $f|_{(-\epsilon, +\epsilon)}(x)$ of $f(x)$ to the interval $(-\epsilon, +\epsilon)$ no matter how small $\epsilon > 0$ is chosen. However $f(x)$ is weakly invertible near 0 in the sense of Theorem 1.1.

Another example is provided by the function

\[ f(x) = \begin{cases} x + x^2 & \text{if } x \text{ is irrational,} \\ x - x^2 & \text{if } x \text{ is rational.} \end{cases} \]

Here $f'(0) = 1$ and $f(x)$ is discontinuous for $x \neq 0$. Again $f(x)$ is weakly invertible near 0 in the sense of Theorem 1.1.

It is the purpose of this note to give estimates on the value of $\epsilon$ above. That is to say we seek to estimate the size of the domain of weak invertibility of $f$ near $a$. Clearly if no additional information besides (1.1) is provided, no estimates on $\epsilon$ can be deduced.

The main result of this note is

**Theorem 1.2.** Let $f : E \to E$ be a mapping from the normed space $E$ into itself. Let the derivative $D$ at $a \in E$ be a bounded isomorphism on $E$ with bounded inverse $D^{-1}$. Assume, in addition, that there exist constants $C$, $r$, $\alpha > 0$ such that

\[ \|f(a + h) - f(a) - Dh\| \leq C\|h\|^{1+\alpha} \]

for $\|h\| \leq r$. Then

\[ 0 < \|h\| < \min \left( r, \left( \frac{1}{C\|D^{-1}\|^{1/\alpha}} \right)^{1/\alpha} \right) \]


implies

(1.4) \[ f(a + h) \neq f(a). \]

2. PROOF OF THEOREM 1.2

Recall that for a bounded linear isomorphism \( D \) on the normed space \( E \) with bounded inverse \( D^{-1} \) we have

(2.1) \[ \|D^{-1}\|^{-1} \|h\| \leq \|D\| \|h\|. \]

Moreover \( \|D^{-1}\|^{-1} \) is the supremum of the constants \( C \) for which the inequality \( C \|h\| \leq \|D\| \|h\| \) holds. In the finite dimensional case \( \|D^{-1}\|^{-1} \) and \( \|D\| \) are, respectively, the smallest and biggest singular values of \( D \), as is well known.

We first analyse the situation when the constraint \( \|h\| \leq r \) is removed from relation (1.2).

Relation (1.2) implies

(2.2) \[ \|f(a + h) - f(a)\| \geq \|D\| \|h\| + C\|h\|^{1+\alpha}. \]

Now let \( 0 < \beta < 1 \) and let \( h \) be such that

(2.3) \[ \|h\| \leq \left( \frac{1 - \beta}{C} \|D^{-1}\|^{-1} \right)^{1/\alpha}. \]

Then relations (2.1) and (2.2) imply, for those values of \( h \) for which (2.3) holds,

(2.4) \[ \|f(a + h) - f(a)\| \geq \beta \|D^{-1}\|^{-1} \|h\|. \]

Because \( \beta \) may be chosen arbitrarily close to 0, we see that

(2.5) \[ \|h\| < \left( \frac{1}{C} \|D^{-1}\|^{-1} \right)^{1/\alpha} \]

implies

(2.6) \[ \|f(a + h) - f(a)\| \geq \gamma \|D^{-1}\|^{-1} \|h\| \]

for some \( \gamma = \gamma_h > 0 \). Hence

(2.7) \[ 0 < \|h\| < \left( \frac{1}{C} \|D^{-1}\|^{-1} \right)^{1/\alpha} \]

implies

(2.8) \[ \|f(a + h) - f(a)\| > 0. \]
We now reinstate the constraint $\|h\| \leq r$ in relation (1.2) and see that
\[
0 < \|h\| < \min \left( r, \left( \frac{1}{C}\|D^{-1}\|^{-1} \right)^{1/\alpha} \right)
\]
implies $\|f(a+h) - f(a)\| > 0$.

**Remark.** In the case where relation (1.2) is replaced by (1.1), the above proof shows that $\|f(a+h) - f(a)\| > 0$ if $\|h\| > 0$ is small enough. Hence we recover, in a more general setting, the results of [1] and [2]; however, no estimates can now be given on the size of $\|h\|$.

For convenience we adapt below the proof of Theorem 1.2 to the setting of the above Remark. Proposition 2.1 is a restatement, in a slightly different form, of Theorem 1.1.

**Proposition 2.1.** Let $f : E \to E$ be a function on the normed vector space $E$. Assume that there exists a bounded linear operator $D : E \to E$ with bounded inverse such that
\[
(2.9) \quad \lim_{h \to 0} \frac{\|f(a+h) - f(h) - Dh\|}{\|h\|} = 0.
\]
Then $f(x)$ is weakly locally invertible near $a$ in the sense that, if $\epsilon > 0$ is small enough, then
\[
0 < \|h\| < \epsilon \quad \text{implies} \quad f(a+h) \neq f(a).
\]

**Proof.** Pick $\epsilon > 0$ small enough so that $\|h\| < \epsilon$ implies
\[
(2.10) \quad \|f(a+h) - f(h) - Dh\| \leq \frac{1}{2}\|D^{-1}\|^{-1}\|h\|.
\]
This is possible because it follows from (2.9) that
\[
(2.11) \quad \|f(a+h) - f(h) - Dh\| = o(\|h\|)
\]
and that
\[
(2.12) \quad \|D^{-1}\|^{-1} > 0
\]
because of the boundedness of $D$ and $D^{-1}$. For those values of $h$ with $o(\|h\|)$ given by (2.11), we now have as a consequence of (2.10) and (2.11)
\[
\|f(a+h) - f(a)\| \geq \|Dh\| - o(\|h\|) \\
\geq \|D^{-1}\|^{-1}\|h\| - o(\|h\|) \\
\geq \frac{1}{2}\|D^{-1}\|^{-1}\|h\|.
\]
Hence, in view of relation (2.12), $0 < \|h\| < \epsilon$ implies $f(a+h) \neq f(a)$.

We reiterate the fact that in the above proof, as well as in the proof of Theorem 1.2, neither the completeness of $E$ nor the continuity of $f$ in a neighborhood of $a$ is assumed.
3. A STABILITY RESULT

Examination of the proof of Theorem 1.2 shows that we also proved the following stability result:

**Proposition 3.1.** Let $f$, $D$, $C$, $\alpha$ be as in Theorem 1.2. Let inequality (1.2) hold. Let $0 < \beta < 1$ and let $h$ be such that

$$
\|h\| \leq \left(\frac{1 - \beta}{C}\|D^{-1}\|^{-1}\right)^{1/\alpha}.
$$

Then, for those values of $h$ for which the above inequality holds,

$$
\|f(a + h) - f(a)\| \geq \beta\|D^{-1}\|^{-1}\|h\|.
$$

Here it is assumed that the constraint $\|h\| < r$ is not present.

Thus Proposition 3.1 is quantitative in nature: it gives estimates on the discrepancy $\|f(a + h) - f(a)\|$ at the expense of shrinking the radius $\left(\frac{1}{C}\|D^{-1}\|^{-1}\right)^{1/\alpha}$ of the ball of center $a$ where $f(x)$ is weakly invertible. In the case where the constraint $\|h\| < r$ is reinstated, the radius under consideration in Theorem 1.2, namely $\min\left(r, \left(\frac{1 - \beta}{C}\|D^{-1}\|^{-1}\right)^{1/\alpha}\right)$, must now be reduced to $\min\left(r, \left(\frac{1 - \beta}{C}\|D^{-1}\|^{-1}\right)^{1/\alpha}\right)$.

Such an estimate of the discrepancy $\|f(a + h) - f(a)\|$ is not present (and cannot be provided) in the general framework of Theorem 1.1. Indeed, as shown below, $\|f(a + h) - f(a)\| > 0$ may become as small as we want if $\|h\|$ is chosen close enough to $\left(\frac{1}{C}\|D^{-1}\|^{-1}\right)^{1/\alpha}$.

Proposition 3.1 is a typical stability result: the bigger the constant $\beta\|D^{-1}\|^{-1}$ is in $\|f(a + h) - f(a)\| \geq \beta\|D^{-1}\|^{-1}\|h\|$, the more stable is the mapping $f$. The concept of stability, in different contexts, is considered in the fundamental paper [4].

4. OPTIMALITY OF THEOREM 1.2 AND PROPOSITION 3.1

It is the purpose of this section to examine, in a simple case, the optimality of Theorem 1.2 and of Proposition 3.1.

We examine first the optimality of Theorem 1.2.

Let $f(x) = x^2$, $a = 1$. Then, noticing that here $C = 1$, $\alpha = 1$, $r = \infty$, we have $D = D_1 = [2]$ and $\|D^{-1}\|^{-1} = 2$. We have $f(1 + h) - f(1) - 2h = h^2$. Theorem 1.1 says that

$$
0 < |h| < 2 \implies f(1) \neq f(1 + h)
$$

which, of course, is the case. Moreover the above implication fails for $|h| \leq 2$. 

We remark that the function \( f(x) \) is not invertible for \(|h| < 2\), but that only (4.1) holds.

Let us now examine the optimality of Proposition 3.1 for the same function \( f \) and the same value of \( a = 1 \). We choose \( \beta = 1/2 \) so that

\[ |h| \leq \frac{1}{2} \|D^{-1}\|^{-1} = 1 \]

and

\[ |f(1 + h) - f(1)| = |2h + h^2|. \]

We see that, indeed, \(|2h + h^2| > |h|\) for \( |h| < 1 \) and that this estimate does not hold for \(|h| \leq 1\).

REFERENCES