NOTES ON MAXIMAL EXCEPTIONAL GRAPHS

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An exceptional graph is a connected graph with least eigenvalue greater than or equal to \(-2\) which is not a generalized line graph. There are finitely many exceptional graphs. Maximal exceptional graphs have been recently identified. In this paper we discuss some details related to the construction of maximal exceptional graphs.

An exceptional graph is a connected graph with least eigenvalue greater than or equal to \(-2\) which is not a generalized line graph. The set \(E\) of exceptional graphs is finite. The number of exceptional graphs is very big but not known. The set \(E\) can be partially ordered by the induced subgraph relation. In this order \(E\) has 20 minimal elements and 473 maximal ones. Minimal exceptional graphs, all having 6 vertices, were found some 25 years ago [3]. Maximal exceptional graphs have been recently identified [4], [5], [6]. This topic and problems related to graphs with least eigenvalue greater than or equal to \(-2\) in general have been presented in the expository article [2] and in the monograph [7]. In this paper we discuss some details related to the construction of maximal exceptional graphs.

There are exactly 473 maximal exceptional graphs. Their number of vertices varies between 22 and 36. They are described in Chapter 6 and Table A6 of [7] where they are denoted by \(G_{001}, G_{002}, \ldots, G_{473}\). These maximal graphs have been constructed by the star complement technique (cf. [7], Chapter 5) starting from exceptional star complements \(H_{001}, H_{002}, \ldots, H_{443}\) (given in Table A2 of [7], all having 8 vertices). Most of the maximal exceptional graphs (432 of them) are 29 vertex cones (the cone over a graph \(S\) is a graph obtained from \(S\) by adding a new vertex adjacent to all vertices of \(S\)) of the form \(G(P)\). Here \(G(P)\) denotes the cone over the graph obtained from \(L(K_8)\) by switching with respect to the edge set of a spanning subgraph \(P\) of \(K_8\).

For all other necessary definitions and previous results the reader is referred to [7].

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1. As the computer search shows (cf. [7], p. 142), the following 9 graphs, in the role of a star complement, give rise to all maximal graphs except for $G_{006}$ (the cone over $L(K_8)$): $H_{424}, H_{425}, H_{431}, H_{433}, H_{435}, H_{436}, H_{437}, H_{439}, H_{440}$.

Other star complements generate a smaller number of maximal graphs. The graph $G_{006}$ has 39 nonisomorphic exceptional star complements.

We shall now formulate a theorem saying that all exceptional graphs can essentially be obtained starting from a single graph, namely $H_{440}$, in the role of star complement for $-2$. This is the main part of Theorem 3.5 of [2].

For convenience, the complement of $H_{440}$ is given in Fig. 1.

Similar theorems can be formulated for each of the other 8 of 9 graphs mentioned above.

**Theorem 1.** Let $G$ be an exceptional graph. Then there exists a graph $H$ such that $G$ is an induced subgraph of $H$ where $H$ has the star complement $H_{440}$ for the least eigenvalue $-2$ or $H$ is the cone over $L(K_8)$.

The theorem follows from the computer search but there is always a tendency in mathematics to prove theorems without recourse to a computer. A sketch of the proof is already given in [2] and here we provide additional details.

**Sketch of the proof (additions).** Let $x$ be a vertex of degree 7 in $H_{440}$. Using brute force we can check that the switching class of $H_{440} - x$ contains line graphs of the 11 graphs of Fig. 2 (on 8 vertices). Corresponding switching sets are indicated in each graph of Fig. 2.

There are three types of edges – thick, thin and doubled; they are represented by thick, thin and doubled lines, respectively, a double line consisting of a thin and of a dotted line. The set of thick edges forms a relevant switching set; also the set of thin edges is a relevant switching set. These two switching sets in any of graphs of Fig. 2 will be denoted by $a$ and $b$ while we refer to graphs by their numbers. In the first case the double edge is considered as thin and in the second case as thick. Thick and thin edges can exchange the roles what could be denoted, say, by an asterisk (for example, $7a^*$) but this will not be used. Partitions of the edge set of a graph into the sets of thick and thin edges can be viewed as colourings of edges by two colours.

We have to prove that $H_{440}$ is an induced subgraph of all maximal exceptional graphs except for $G_{006}$.

Consider a maximal exceptional graph $G$ of the form $G(P)$ for some 8 vertex graph $P$. (For other maximal exceptional graphs the matter was already settled in [2]). Suppose that $G$ contains an induced subgraph $E$ isomorphic to $H_{440}$. Let $x$ be the vertex of $G$ whose degree in $E$ is equal to 7. Let us assume that the vertex $x$ coincides with the top of the cone $G(P)$. Then $E - x = H_{440} - x$ is a subgraph...
of $G(P) - x$. It is switching equivalent to the line graph $L(F)$ of a graph $F$ on 8 vertices. The graph $F$ and the switching partition $F_1 \cup F_2$ of the edge set of $F$ which yields $E - x$ should coincide with one of the cases in Fig. 2. For any $P$ we must have either $F_1 \subset P$ and $F_2 \subset \overline{P}$ or $F_1 \subset \overline{P}$ and $F_2 \subset P$.

We can represent graph $P$ by thick edges and its complement $\overline{P}$ by thin ones. By a RAMSEY type argument either $P$ or $\overline{P}$ contains a triangle.
Suppose $P$ has a (thick) triangle $a, b, c$ and all other edges incident to the vertices of the triangle are thin. The remaining five vertices contain a monochromatic path $P_4$, say $d, e, f, g$. If it is thick, then the edges $ab, ch, cd, de, ef, fg$ give rise to the situation 1a from Fig. 2 and we are done. If $P_4$ is thin, then the edges $ab, bc, ac, de, ef, fg$ provide 5a.

Next suppose that just one extra thick edge $cd$ is incident to one of the vertices of the thick triangle and that other edges incident to $a, b, c, d$ are thin. Suppose first that the remaining 4 vertices $e, f, g, h$ are mutually joined by at least one thin edge, say $ef$. Then the edges $ab, ac, cd, de, ef$ give the case 5b. Otherwise the edges $ge, gf, gh$ are thick and together with thin edges $ad, bd, de, ce$ give rise to 2a.

In the same spirit we can continue considering the cases and complete the proof of the assertion.

However, the author has not succeeded in finding a short classification of possible cases.

2. Next, we are interested in how the line graphs $L(K_{m,n})$ of the complete bipartite graphs $K_{m,n}$ $(m, n > 1; m + n = 9)$ can be represented in root systems. Since $L(K_{m,n}) = K_m + K_n$, the spectrum of this graph is $m + n - 2, (m - 2)^{n-1}, (n - 2)^{m-1}, (-2)^{(m-1)(n-1)}$, multiplicities of eigenvalues being represented as exponents. The number of eigenvalues greater than $-2$ is $1 + n - 1 + m - 1 = m + n - 1 = 8$. Hence, all these graphs can be represented in the exceptional root system $E_8$.

The graphs in question, namely $K_{5,4}, K_{6,3}$ and $K_{7,2}$, are not subgraphs of $K_8$ and therefore it is not obvious how their line graphs can be embedded into a graph of the form $G(P)$. However, they are induced subgraphs of some of the 473 maximal exceptional graphs. Here we construct such an inclusion.

We use a representation of $E_8$ in the 8-dimensional subspace of $\mathbb{R}^8$ determined by $\sum_{i=1}^9 x_i = 0$ (cf. [1]):

$$3e_i - 3e_j \ (i, j = 1, 2, \ldots, 9; \ i < j) \ \binom{9}{2} = 36 \text{ vectors},$$

$$\epsilon - 3e_i - 3e_j - 3e_k \ (i, j, k = 1, 2, \ldots, 9; \ i < j < k) \ \binom{9}{3} = 84 \text{ vectors}.$$ 

Here $\epsilon$ denotes the all-1 vector and vectors $e_1, e_2, \ldots, e_9$ represent the standard basis of $\mathbb{R}^8$.

**Proposition 2.** Maximal exceptional graph $G471$ contains $L(K_{5,4})$ as an induced subgraph.

**Proof.** $L(K_{5,4})$ can be represented by the 20 vectors $3e_i - 3e_j \ (i = 1, 2, \ldots, 5; \ j = 6, 7, 8, 9)$. We can add 4 vectors $\epsilon - 3e_7 - 3e_8 - 3e_9, \epsilon - 3e_6 - 3e_8 - 3e_9, \epsilon - 3e_6 - 3e_7 - 3e_9$, and $\epsilon - 3e_6 - 3e_7 - 3e_8$. In addition, 10 vectors $3e_i + 3e_j - \epsilon \ (i, j = 1, 2, \ldots, 5; \ i < j)$ can be added, and thereafter no others. In this way we obtain a maximal exceptional graph on 34 vertices. It is $G471$. 

REFERENCES


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