COMPLETE MONOTONICITY PROPERTIES
FOR A RATIO OF GAMMA FUNCTIONS

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Define for \( x > 0 \)
\[
F(x) = \frac{\Gamma(2x)}{x \Gamma^2(x)} \quad \text{and} \quad G(x) = \frac{\Gamma(2x)}{\Gamma^2(x)}.
\]

In this paper, we consider the logarithmically complete monotonicity properties for the functions \( F \) and \( 1/G \).

The gamma function is defined for \( \text{Re} \, z > 0 \) by
\[
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt.
\]

The psi or digamma function, the logarithmic derivative of the gamma function, can be expressed as
\[
\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-zt}}{1 - e^{-t}} \, dt,
\]
where \( \gamma = 0.57721566490153286 \ldots \) is the Euler-Mascheroni constant.

In 1997, Merkle [1] showed that the function \( F(x) = \frac{\Gamma(2x)}{x \Gamma^2(x)} \) is strictly log-convex and the function \( G(x) = \frac{\Gamma(2x)}{\Gamma^2(x)} \) is strictly log-concave on \((0, \infty)\). In this paper, we extend the results given by Merkle; we consider the logarithmically complete monotonicity properties for the functions \( F \) and \( 1/G \). Recall that a function \( f \) is said to be completely monotonic on an interval \( I \), if \( f \) has derivatives of all orders on \( I \) and satisfies
\[
(-1)^n f^{(n)}(x) \geq 0 \quad (x \in I; n = 0, 1, 2, \ldots).
\]

If the inequality (3) is strict, then \( f \) is said to be strictly completely monotonic on \( I \).

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**Definition.** A positive function $f$ is said to be logarithmically completely monotonic on an interval $I$ if its logarithm $\ln f$ satisfies

\[ (-1)^n \left( \ln f(x) \right)^{(n)} \geq 0 \quad (x \in I; n = 1, 2, \ldots). \]

If inequality (4) is strict for all $x \in I$ and for all $n \geq 1$, then $f$ is said to be strictly logarithmically completely monotonic.

This definition was introduced in [2] by F. Qi and B.-N. Guo. Moreover, the authors showed that a (strictly) logarithmically completely monotonic function must be (strictly) completely monotonic.

The purpose of this paper is to establish the following result.

**Theorem.** Let $I = (0, +\infty)$ and let $F(x) = \frac{\Gamma(2x)}{x\Gamma^2(x)}$, $G(x) = \frac{\Gamma(2x)}{\Gamma^2(x)}$, $x \in I$.

Then we have

(A) $\left( \ln F(x) \right)' > 0$, $x \in I$,

(B) $(-1)^n \left( \ln F(x) \right)^{(n)} > 0$ for $x \in I$ and $n = 2, 3, \ldots$,

(C) The function $1/G$ is strictly logarithmically completely monotonic on $I$.

**Proof.** Using the duplication formula and the translation formula for the gamma function

\[ \Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x)\Gamma(x+1/2), \]

\[ \Gamma(x+1) = x\Gamma(x), \]

we conclude that

\[ F(x) = \frac{2^{2x-1}\Gamma(x+1/2)}{\sqrt{\pi}\Gamma(x+1)}. \]

Taking logarithm and differentiation yields

\[ \left( \ln F(x) \right)' = 2\ln 2 + \psi(x+1/2) - \psi(x+1) \]

\[ = 2\ln 2 + \int_0^\infty \frac{e^{-(x+1)t} - e^{-(x+1/2)t}}{1 - e^{-t}} \, dt \]

\[ = 2\ln 2 - \int_0^\infty \frac{e^{-xt}}{1 + e^{t/2}} \, dt \]

and therefore

\[ (-1)^n \left( \ln F(x) \right)^{(n)} = \int_0^\infty \frac{t^{n-1}}{1 + e^{t/2}} e^{-xt} \, dt > 0 \quad (x > 0; \ n = 2, 3, \ldots). \]

Clearly, $\left( \ln F(x) \right)'' > 0$, and then the function $x \mapsto \left( \ln F(x) \right)'$ is strictly increasing.
on \((0, \infty)\), which implies for \(x > 0\)

\[
\left( \ln F(x) \right)' > \left( \ln F(x) \right)'_{x=0} = 2 \ln 2 - \int_0^\infty \frac{1}{1 + e^{t/2}} \, dt
\]

\[
= 2 \ln 2 + 2 \int_0^\infty \frac{1}{1 + e^{-t/2}} \, d(1 + e^{-t/2}) = 0.
\]

Using (5) and (6) we conclude that

\[
G(x) = \frac{2^{2x-1} \Gamma(x + 1/2)}{\sqrt{\pi} \Gamma(x)}.
\]

Taking logarithm and differentiation yields

\[
\left( \ln \left( 1/G(x) \right) \right)' = -2 \ln 2 - \psi(x + 1/2) + \psi(x)
\]

\[
= -2 \ln 2 - \int_0^\infty \frac{e^{-xt} - e^{-(x+1/2)t}}{1 - e^{-t}} \, dt
\]

\[
= -2 \ln 2 - \int_0^\infty \frac{e^{-xt}}{1 + e^{-t/2}} \, dt < 0
\]

and therefore

\[
(-1)^n \left( \ln \left( 1/G(x) \right) \right)^{(n)} = \int_0^\infty \frac{t^{n-1}}{1 + e^{-t/2}} e^{-xt} \, dt > 0 \quad (x > 0; \ n = 2, 3, \ldots).
\]

The proof is complete.

**REFERENCES**


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