ON VOLUMES OF $n$-DIMENSIONAL PARALLELEPIPEDS IN $\ell^p$ SPACES

H. Gunawan, W. Setya-Budhi, Mashadi, S. Gemawati

Given a linearly independent set of $n$ vectors in a normed space, we are interested in computing the “volume” of the $n$-dimensional parallelepiped spanned by them. In $\ell^p$ ($1 \leq p < \infty$), we can use the known semi-inner product and obtain, in general, $n!$ ways of doing it, depending on the order of the vectors. We show, however, that all resulting “volumes” satisfy one common inequality.

1. INTRODUCTION

On a normed space $(X, \| \cdot \|)$, the functional $g : X^2 \to \mathbb{R}$ defined by the formula

$$g(x, y) := \frac{\|x\|}{2} (\lambda_+(x, y) + \lambda_-(x, y)),$$

where

$$\lambda_{\pm}(x, y) := \lim_{t \to \pm \infty} t^{-1} (\|x + ty\| - \|x\|),$$

satisfies the following properties:

(a) $g(x, x) = \|x\|^2$ for all $x \in X$;
(b) $g(\alpha x, \beta y) = \alpha \beta g(x, y)$ for all $x, y \in X$, $\alpha, \beta \in \mathbb{R}$;
(c) $g(x, x + y) = \|x\|^2 + g(x, y)$ for all $x, y \in X$;
(d) $|g(x, y)| \leq \|x\| \|y\|$ for all $x, y \in X$.

If, in addition, the functional $g(x, y)$ is linear in $y \in X$, it is called a semi-inner product on $X$ (see [3, 4]). For instance, the functional

$$g(x, y) := \|x\|^{2-p} \sum_k |x_k|^{p-1} \text{sgn}(x_k) y_k, \quad x = (x_k), \ y = (y_k) \in \ell^p,$$

2000 Mathematics Subject Classification: 46B20, 46B45, 46C50, 46C99

Keywords and Phrases: $n$-dimensional parallelepipeds, semi-inner products, orthogonal projection in normed spaces, $n$-norms, $\ell^p$ spaces
defines a semi-inner product on the space \( \ell^p \) of \( p \)-summable sequences of real numbers, for \( 1 \leq p < \infty \). (Here \( \| \cdot \|_p \) is the usual norm on \( \ell^p \).)

Using a semi-inner product \( g \), one may define the notion of orthogonality on \( X \). In particular, we can define

\[ x \perp_g y \Leftrightarrow g(x, y) = 0. \]

(Note that since \( g \) is in general not commutative, \( x \perp_g y \) does not imply that \( y \perp_g x \).) Further, one can also define the \( g \)-orthogonal projection of \( y \) on \( x \) by

\[ y_x := \frac{g(x, y)}{\|x\|^2} x, \]

and call \( y - y_x \) the \( g \)-orthogonal complement of \( y \) on \( x \). Notice here that \( x \perp_g y - y_x \).

In general, given a vector \( y \in X \) and a subspace \( S = \text{span} \{x_1, \ldots, x_k\} \) of \( X \) with \( \Gamma(x_1, \ldots, x_k) := \det(g(x_i, x_j)) \neq 0 \), we can define the \( g \)-orthogonal projection of \( y \) on \( S \) by

\[ y_S := \frac{1}{\Gamma(x_1, \ldots, x_k)} \begin{vmatrix} 0 & x_1 & \cdots & x_k \\ g(x_1, y) & g(x_1, x_1) & \cdots & g(x_1, x_k) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_k, y) & g(x_k, x_1) & \cdots & g(x_k, x_k) \end{vmatrix}, \]

for which its orthogonal complement \( y - y_S \) is given by

\[ y - y_S = \frac{1}{\Gamma(x_1, \ldots, x_k)} \begin{vmatrix} y & x_1 & \cdots & x_k \\ g(x_1, y) & g(x_1, x_1) & \cdots & g(x_1, x_k) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_k, y) & g(x_k, x_1) & \cdots & g(x_k, x_k) \end{vmatrix}. \]

Observe here that \( x_i \perp_g y - y_S \) for each \( i = 1, \ldots, k \).

Next, given a finite sequence of linearly independent vectors \( x_1, \ldots, x_n \) \((n \geq 2)\) in \( X \), we can construct a left \( g \)-orthogonal sequence \( x_1^*, \ldots, x_n^* \) as in [4]:

Put \( x_1^* := x_1 \) and, for \( i = 2, \ldots, n \), let

\[ x_i^* := x_i - (x_i)_{S_{i-1}}, \]

where \( S_{i-1} = \text{span} \{x_1^*, \ldots, x_{i-1}^*\} \). Then clearly \( x_i^* \perp_g x_j^* \) for \( i, j = 1, \ldots, n \) with \( i < j \). Having done so, we may now define the “volume” of the \( n \)-dimensional parallelepiped spanned by \( x_1, \ldots, x_n \) in \( X \) to be

\[ V(x_1, \ldots, x_n) := \prod_{i=1}^{n} \|x_i^*\|. \]

Due to the limitation of \( g \), however, \( V(x_1, \ldots, x_n) \) may not be invariant under permutations of \( (x_1, \ldots, x_n) \).
In the following section, we shall consider the parallelepipeds spanned by $n$ linearly independent vectors in $\ell^p$ ($1 \leq p < \infty$). Our main result shows that their "volumes" satisfy one common inequality, which involves the natural $n$-norm of those vectors in $\ell^p$.

2. MAIN RESULT

Suppose, hereafter, that $1 \leq p < \infty$. The so-called (natural) $n$-norm on $\ell^p$ is the functional $\|\cdot, \ldots, \cdot\|_p : (\ell^p)^n \to \mathbb{R}$ defined by the formula

$$
\|x_1, \ldots, x_n\|_p := \left( \frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} \left| \begin{array}{ccc}
x_{1j_1} & \cdots & x_{1j_n} \\
\vdots & \ddots & \vdots \\
x_{nj_1} & \cdots & x_{nj_n}
\end{array} \right|_p \right)^{1/p}
$$

(see [1]). (Here the outer $| \cdots |$ denotes the absolute value, while the inner $| \cdots |$ denotes the determinant.) For $p = 2$, we have $\|x_1, \ldots, x_n\|_2 = \sqrt{\text{det}(\langle x_i, x_j \rangle)}$, which represents the Euclidean volume of the $n$-dimensional parallelepiped spanned by $x_1, \ldots, x_n$ in $\ell^2$. (Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\ell^2$.) For $n = 1$, the 1-norm coincides with the usual norm on $\ell^p$. The $n$-norm $\|\cdot, \ldots, \cdot\|_p$ on $\ell^p$ satisfies the following four basic properties:

(a) $\|x_1, \ldots, x_n\|_p = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent;
(b) $\|x_1, \ldots, x_n\|_p$ is invariant under permutation;
(c) $\|\alpha x_1, x_2, \ldots, x_n\|_p = |\alpha| \|x_1, x_2, \ldots, x_n\|_p$ for any $\alpha \in \mathbb{R}$;
(d) $\|x_1 + x_1', x_2, \ldots, x_n\|_p \leq \|x_1, x_2, \ldots, x_n\|_p + \|x_1', x_2, \ldots, x_n\|_p$.

Further properties of this functional on $\ell^p$ can be found in [1]. See also [2, 5], and the references therein, for related works.

Our theorem below relates the "volume" $V(x_1, \ldots, x_n)$ defined by (3) and the $n$-norm $\|x_1, \ldots, x_n\|_p$, which also represents a volume of the $n$-dimensional parallelepiped spanned by $x_1, \ldots, x_n$ in $\ell^p$.

We assume hereafter that $n \geq 2$.

**Theorem 1.** Let $\{x_1, \ldots, x_n\}$ be a linearly independent set of vectors in $\ell^p$. For any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$, define $V(x_{i_1}, \ldots, x_{i_n})$ as in (3) by using the semi-inner product $g$ in (1), with $x_{i_1}^* = x_{i_1}$ and so forth as in (2). Then we have

$$
V(x_{i_1}, \ldots, x_{i_n}) \leq (n!)^{1/p}\|x_{i_1}, \ldots, x_{i_n}\|_p.
$$

The following example illustrates the situation in $\ell^3$. Let $x_1 = (1, 0, 0, \ldots)$ and $x_2 = (1, 1, 0, \ldots)$. Put $x_1^* = x_1$ and $x_2^* = x_2 - (x_2)_{x_1} = (0, 1, 0, \ldots)$. Then we have $V(x_1, x_2) = \|x_1^*\|_1\|x_2^*\|_1 = 1 \cdot 1 = 1$. But if we put $x_1^* = x_2$ and $x_2^* = x_1 - (x_1)_{x_2} = (\frac{1}{2}, -\frac{1}{2}, 0, \ldots)$, then we have $V(x_2, x_1) = \|x_1^*\|_1\|x_2^*\|_1 = 2 \cdot 1 = 2$.

Meanwhile,
\[ ||x_1, x_2||_1 = \frac{1}{2} \sum_j \sum_k || x_{1j} x_{2k} || = \frac{1}{2} \left( \left| \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right| + \left| \begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right| \right) = \frac{1}{2} (1 + 1) = 1. \]

Hence we see that \( V(x_1, x_2) \leq 2||x_1, x_2||_1 \) for each permutation \((i_1, i_2)\) of \((1, 2)\).

**Proof of Theorem 1.** Since \( ||x_1, \ldots, x_n||_p \) is invariant under permutation, it suffices for us to show that

\[
V(x_1, \ldots, x_n) \leq (n!)^{1/p} ||x_1, \ldots, x_n||_p.
\]

Recall that \( V(x_1, \ldots, x_n) = \prod_{i=1}^n ||x_i^*||_p \), where \( x_1^*, \ldots, x_n^* \) is the left \( g \)-orthogonal sequence constructed from \( x_1, \ldots, x_n \) (with \( x_1^* = x_1 \) and so forth as in (2)). From the construction of \( x_1^*, \ldots, x_n^* \), we have

\[
x_n^* = \frac{1}{\Gamma(x_1^*, \ldots, x_{n-1}^*)} \left| \begin{array}{cccc} x_n & x_1^* & \cdots & x_{n-1}^* \\ g(x_1^*, x_n) & g(x_1^*, x_1^*) & \cdots & g(x_1^*, x_{n-1}^*) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_{n-1}^*, x_n) & g(x_{n-1}^*, x_1^*) & \cdots & g(x_{n-1}^*, x_{n-1}^*) \end{array} \right|.
\]

But \( \Gamma(x_1^*, \ldots, x_{n-1}^*) = \prod_{i=1}^{n-1} ||x_i^*||_p^2 \), and so

\[
||x_n^*||_p = \prod_{i=1}^{n-1} ||x_i^*||_p^{-2} \left( \sum_{j_n} \left| \begin{array}{cccc} x_{nj_n} & x_{1j_n}^* & \cdots & x_{n-1,j_n}^* \\ g(x_1^*, x_{nj_n}) & g(x_1^*, x_{1j_n}^*) & \cdots & g(x_1^*, x_{n-1,j_n}^*) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_{n-1}^*, x_{nj_n}) & g(x_{n-1}^*, x_{1j_n}^*) & \cdots & g(x_{n-1}^*, x_{n-1,j_n}^*) \end{array} \right| \right)^{1/p}.
\]

Hence, the “volume” \( V(x_1, \ldots, x_n) \) is equal to

\[
\prod_{i=1}^{n-1} ||x_i^*||_p^{-1} \left( \sum_{j_n} \left| \begin{array}{cccc} x_{nj_n} & x_{1j_n}^* & \cdots & x_{n-1,j_n}^* \\ g(x_1^*, x_{nj_n}) & g(x_1^*, x_{1j_n}^*) & \cdots & g(x_1^*, x_{n-1,j_n}^*) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_{n-1}^*, x_{nj_n}) & g(x_{n-1}^*, x_{1j_n}^*) & \cdots & g(x_{n-1}^*, x_{n-1,j_n}^*) \end{array} \right| \right)^{1/p}.
\]

By using properties of determinants, we find that \( V(x_1, \ldots, x_n) \) is equal to

\[
\left| \begin{array}{cccc} g(x_1^*, x_1^*) & \cdots & g(x_1^*, x_{n-1}^*) & x_{1j_n} \\ \vdots & \ddots & \vdots & \vdots \\ g(x_{n-1}^*, x_1^*) & \cdots & g(x_{n-1}^*, x_{n-1}^*) & x_{n-1,j_n} \\ g(x_1^*, x_n) & \cdots & g(x_{n-1}^*, x_n) & x_{nj_n} \end{array} \right|^{1/p}.
\]
Since \( x_1^* = x_1 \) and \( g(x, y) \) is linear in \( y \), it follows that \( V(x_1, \ldots, x_n) \) is equal to

\[
\left( \sum_{j_n} \prod_{i=1}^{n-1} ||x_i^*||_p^{-1} \begin{vmatrix} g(x_1^*, x_1) & \cdots & g(x_{n-1}^*, x_1) & x_{1j_n} \\ \vdots & \ddots & \vdots & \vdots \\ g(x_1^*, x_{n-1}) & \cdots & g(x_{n-1}^*, x_{n-1}) & x_{n-1, j_n} \\ g(x_1^*, x_n) & \cdots & g(x_{n-1}^*, x_n) & x_{nj_n} \end{vmatrix} \right)^{1/p}.
\]

Now \( g(x_i^*, x_k) = ||x_i^*||_p^{2-p} \sum_j |x_{ij}|^{p-1} \text{sgn}(x_{ij}) x_{kj} \), and we can take the sums out of the determinant, so that the above expression is dominated by

\[
\left( \sum_{j_n} \left( \sum_{j_{n-1}} \cdots \sum_{j_1} \frac{|x_{n-1,j_{n-1}}|^{p-1}}{||x_{n-1}||_p^{p-1}} \cdots \frac{|x_{1j_1}|^{p-1}}{||x_1||_p^{p-1}} \left| \begin{vmatrix} x_{1j_1} & \cdots & x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \cdots & x_{nj_n} \end{vmatrix} \right| \right)^{1/p} \right).
\]

By Hölder's inequality (applied to the multiple series inside the inner square brackets), the last expression is dominated by

\[
\left( \sum_{j_n} \left( \sum_{j_{n-1}} \cdots \sum_{j_1} \left| \begin{vmatrix} x_{1j_1} & \cdots & x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \cdots & x_{nj_n} \end{vmatrix} \right| \right)^{1/p} \right).
\]

which is equal to \((nl)^{1/p} ||x_1, \ldots, x_n||_p\). This proves our theorem.

3. CONCLUDING REMARKS

Unlike in inner product spaces, we generally do not have an analogue of Hadamard's inequality (see, e.g., [6, p. 597])

\[
V(x_1, \ldots, x_n) \leq \prod_{i=1}^n ||x_i||.
\]

For a counterexample, take \( x_1 = (1, 2, 0, \ldots) \) and \( x_2 = (2, -1, 0, \ldots) \) in \( \ell^1 \). Then one may check that \( V(x_1, x_2) = V(x_2, x_1) = 3 \cdot \frac{10}{11} > ||x_1||_1 ||x_2||_1 \). (This adds a reason why we write the word “volume” between quotation marks for \( V(x_1, \ldots, x_n) \).

It is worth noting, however, that the analogue of Hadamard’s inequality is satisfied particularly by the \( n \)-norm \( ||, \ldots, ||_1 \) on \( \ell^1 \). Indeed, the inequality

\[
||x_1, \ldots, x_n||_1 \leq \prod_{i=1}^n ||x_i||_1
\]

holds for every \( x_1, \ldots, x_n \) in \( \ell^1 \) (see [1]). Hence the \( n \)-norm \( ||, \ldots, ||_1 \) has the desirable properties for volumes of \( n \)-dimensional parallelepipeds in \( \ell^1 \).
Volumes of $n$-dimensional parallelepipeds in $\ell^p$ spaces

The reader might also wonder why we do not define the volume of the $n$-dimensional parallelepiped spanned by $x_1, \ldots, x_n$ in $X$ to be

$$V(x_1, \ldots, x_n) := \sqrt{\Gamma(x_1, \ldots, x_n)},$$

instead of (3). Although $\Gamma(x_1, \ldots, x_n) = \det(g(x_i, x_j))$ is invariant under permutation, there are a few problems with this formula. First, $\Gamma(x_1, \ldots, x_n)$ may be negative when $n \geq 3$. For example, take $x_1 = (1, 2, -1/10, 0, \ldots)$, $x_2 = (2, 1, 1/10, 0, \ldots)$, and $x_3 = (1, -1, 1, 0, \ldots)$ in $\ell^1$. Then one may check that $\Gamma(x_1, x_2, x_3) < 0$. Next, for $n = 2$, we can have $\Gamma(x_1, x_2) = 0$ even though $x_1$ and $x_2$ are linearly dependent. For example, take $x_1 = (1, 2, 0, \ldots)$ and $x_2 = (2, 1, 0, \ldots)$ in $\ell^1$. Clearly $x_1$ and $x_2$ are linearly independent. But one may check that $g(x_i, x_j) = 9$ for $i, j = 1, 2$, and so $\Gamma(x_1, x_2) = \begin{vmatrix} 9 & 9 \\ 9 & 9 \end{vmatrix} = 0$. (This explains why we require $\Gamma(x_1, \ldots, x_n) \neq 0$ when we define the $g$-orthogonal projection on the subspace $S = \text{span}\{x_1, \ldots, x_n\}$.)

One should also note that the analogue of Hadamard’s inequality is not satisfied by $|\Gamma|$, that is, the inequality

$$|\Gamma(x_1, \ldots, x_n)| \leq \prod_{i=1}^n \|x_i\|^2$$

does not hold. For a counterexample, take $x_1 = (1, 2, 0, \ldots)$ and $x_2 = (2, -1, 0, \ldots)$ in $\ell^1$. Then we have $|\Gamma(x_1, x_2)| = 90 > \|x_1\|^2 \|x_2\|^2$. Nevertheless, we have the following result for $\Gamma$. (We leave its proof to the reader.)

**Theorem 2.** The inequality

$$|\Gamma(x_1, \ldots, x_n)| \leq (n!)^{1/p} \|x_1, \ldots, x_n\|_p \prod_{i=1}^n \|x_i\|_p$$

holds for every $x_1, \ldots, x_n$ in $\ell^p$.

**Acknowledgement.** The research is supported by the Directorate General of Higher Education of Republic of Indonesia through Hibah Pekerti I Program, 2003/2004.

**REFERENCES**


H. Gunawan, W. Setya-Budhi
Department of Mathematics, Bandung Institute of Technology, Bandung 40132, Indonesia
E-mail: hgunawan@dns.math.itb.ac.id
      wono@dns.math.itb.ac.id

Mashadi, S. Gemawati
Department of Mathematics, University of Riau, Pekanbaru 28293, Indonesia
E-mail: mash-mat@unri.ac.id