ON TWO CONVERGENCE TESTS

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In this note we present the relation between Cauchy’s second test and Raabe–Duhamel’s test.

1. INTRODUCTION

In the theory of series with positive terms we use two elementary convergence tests that depend on the following limits:

I \[ \lim_{n \to +\infty} \frac{\ln(1/a_n)}{\ln n} = \lambda_1 \] (Cauchy’s second test),

II \[ \lim_{n \to +\infty} \left( n \left(1 - \frac{a_{n+1}}{a_n}\right) \right) = \lambda_2 \] (Raabe–Duhamel’s test),

where \( a_n \) is the term of order \( n \) in the series and \( \ln \) denotes natural logarithm.

It is a known result (see [1]) that if \( \lambda_2 \) exists, finite or infinite, the \( \lambda_1 \) exists and \( \lambda_1 = \lambda_2 \). The two tests are then equivalent, the first (I) being more general than the second (II).

The purpose of this note is to extend this result.

2. MAIN RESULT

Theorem. Let \( (a_n) \) be a sequence in \( \mathbb{R} \) such that \( a_n > 0 \) for all \( n \in \mathbb{N} \). Then

\[
\lim_{n \to +\infty} R_n \leq \lim_{n \to +\infty} C_n \leq \lim_{n \to +\infty} C_n \leq \lim_{n \to +\infty} R_n,
\]

where \( R_n := n \left(1 - \frac{a_{n+1}}{a_n}\right) \) and \( C_n := \frac{\ln(1/a_n)}{\ln n} \).

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Proof. The inner inequality is obvious, and the two outer inequalities are so closely similar that we may content with proving one of them. Let us choose the left hand inequality and put $\lim R_n = \ell$.

(i) First, we assume that $\ell$ is finite. Then for arbitrary small $\varepsilon > 0$ exists $n_0(\varepsilon) \in \mathbb{N}$ such that

$$n \left(1 - \frac{a_{n+1}}{a_n}\right) > \ell - \varepsilon = \ell_1, \text{ for all } n \geq n_0.$$ 

Then

$$\ln\left(\frac{1}{a_n}\right) - \ln\left(\frac{1}{a_{n+1}}\right) < \ln \left(1 - \frac{\ell_1}{n}\right).$$

But $\ln(1+x) = x - \frac{1}{2} x^2 M(x)$, where $M(x) > 0$ and is bounded in $-1 < B \leq x \leq A$.

Now we have

$$\ln\left(\frac{1}{a_n}\right) - \ln\left(\frac{1}{a_{n+1}}\right) < -\ell_1 \sum_{i=n_0}^{n_0+k-1} \frac{1}{i} - \frac{1}{2} \ell_1^2 \sum_{i=n_0}^{n_0+k-1} M(i)$$

and therefore we get

$$\frac{\ln(1/a_{n_0+k})}{\ln(n_0+k)} > \frac{\ln(1/a_0)}{\ln(n_0+k)} + \frac{\ell_1}{\ln(n_0+k)} \sum_{i=n_0}^{n_0+k-1} \frac{1}{i} - \frac{1}{2} \frac{\ell_1^2}{\ln(n_0+k)} \sum_{i=n_0}^{n_0+k-1} M(i).$$

Since

$$\sum_{i=n_0}^{n_0+k-1} \frac{1}{i} = H(n_0 + k - 1) - H(n_0 - 1)$$

$$= \ln(n_0 + k - 1) - \ln(n_0 - 1) + o(1), \quad k \to +\infty,$$

where

$$H(n) := 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \ln n + \gamma + o(1), \quad n \to +\infty,$$

$\gamma$ is Euler’s constant and the sum $\sum_{i=n_0}^{n_0+k-1} \frac{M(i)}{i^2}$ is bounded, letting $k \to +\infty$ from

(2) follows that

$$\lim_{k \to +\infty} \frac{\ln(1/a_{n_0+k})}{\ln(n_0+k)} \geq \ell_1 = \ell - \varepsilon, \quad \text{for all } \varepsilon > 0,$$

i.e. we obtain $\lim C_n \geq \ell$. 

(ii) If $\ell = +\infty$, we have immediately that $\lim_{n \to +\infty} R_n = +\infty$. Then for arbitrary large $E > 0$ exists $n_0(E) \in \mathbb{N}$, such that

$$n \left(1 - \frac{a_{n+1}}{a_n}\right) > E, \quad \text{for all } n \geq n_0(E).$$

Now we obtain similarly as in (i)

$$\lim \frac{\ln(1/a_n)}{\ln n} > E.$$

Since $E > 0$ is arbitrary large, it follows that $\lim_{n \to +\infty} C_n = +\infty$.

(iii) If $\ell = -\infty$, then $\lim R_n \leq \lim C_n$ is trivial truth.

Remark. In the relation (1) strictly inequalities can be hold. Let us consider the sequence $(a_n)$:

$$a_n = \begin{cases} 1/(2k), & n = 2k \\ 1/(2k + 1/k), & n = 2k + 1 \end{cases} \quad (k = 1, 2, \ldots).$$

Then $\lim R_n = 0$, $\lim R_n = 2$ but $\lim C_n = \lim C_n = 1$.

Finally, we remark that the relation (1) is similar with

$$\lim \left(\frac{a_{n+1}}{a_n}\right) \leq \lim \sqrt[n]{a_n} \leq \lim \sqrt[n]{a_n} \leq \lim \left(\frac{a_{n+1}}{a_n}\right),$$

see [2], p. 277.

REFERENCES


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