LOGARITHMICALLY COMPLETE
MONOTONICITY AND SHUR-CONVEXITY
FOR SOME RATIOS OF GAMMA FUNCTIONS

Ai-Jun Li, Wei-Zhen Zhao, Chao-Ping Chen

Define
\[ F(x) = \frac{\Gamma(mx)}{x^{m-1}\Gamma^m(x)} \quad \text{and} \quad G(x) = \frac{\Gamma(mx)}{\Gamma^m(x)} \]
for \( x > 0 \) and \( m = 2, 3, \ldots \). In this paper, we consider the logarithmically complete monotonicity properties for the function \( F \) and \( 1/G \), and we prove that the function
\[ \phi(x) = \prod_{i=1}^{n} \frac{\Gamma(mx_i + 1)}{\Gamma^m(x_i + 1)} \]
is strictly Schur-convex on \((-1/m, +\infty)^n\).

In 1997, Merkle [6] showed that the function \( F(x) = \frac{\Gamma(2x)}{x^2\Gamma^2(x)} \) is strictly log-convex, the function \( G(x) = \frac{\Gamma(2x)}{\Gamma^2(x)} \) is strictly log-concave on \((0, \infty)\) and the function
\[ (1) \quad \phi(x) = \prod_{i=1}^{n} \frac{\Gamma(2x_i + 1)}{\Gamma^2(x_i + 1)} \]
is strictly Schur-convex on \( x = (x_1, \ldots, x_n) \in (-1/2, +\infty)^n \). Recently, Chen [3] has proved that \((-1)^n(\ln F(x))^{(n)}>0\) for \( x \in (0, \infty), n = 2, 3, \ldots \) and the function \( 1/G \) is strictly logarithmically completely monotonic on \((0, \infty)\). Motivated by the results above, we will extend the function \( F(x), G(x) \) to general forms.

2000 Mathematics Subject Classification: 33B15, 26A48

Keywords and Phrases: Gamma function, psi function, (logarithmically) complete monotonic function, Schur-convex.

The authors were supported in part by the Science Foundation of the Project for Fostering Innovation Talents at Universities of Henan Province, China.
We recall that a function $f$ is said to be completely monotonic on an interval $I \subset \mathbb{R}$ if $f$ has derivatives of all orders and satisfies
\[(2) \quad (-1)^n f^{(n)}(x) \geq 0 \quad (x \in I; n = 0, 1, 2, \ldots).\]

If the inequality (2) is strict, then $f$ is said to be strictly completely monotonic on $I$. A detailed collection of the most important properties of completely monotonic functions can be found in [9, Chapter IV], and in an abstract in [2].

A positive function $f$ is said to be logarithmically completely monotonic on an interval $I$ if its logarithm $\log f$ satisfies
\[(3) \quad (-1)^n [\log f(x)]^{(n)} \geq 0 \quad (x \in I, n \in \mathbb{N} = 1, 2, \ldots).\]

If inequality (3) is strict, then $f$ is said to be strictly logarithmically completely monotonic. The inequality (3) is equivalent to the requirement that the function $-(\log f(x))'$ is completely monotonic. Among other things, it is proved in [4, 8] that a (strictly) logarithmically completely monotonic function is always (strictly) completely monotonic, but not conversely.

The classical gamma function defined as
\[
\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad (x > 0)
\]
is one of the most important functions in analysis and its applications.

The psi or digamma function, the logarithmic derivative of the gamma function can be defined [5, p. 16] as
\[
(4) \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}dt \quad (x > 0),
\]
where $\gamma$ is the Euler-Mascheroni constant. The derivatives of $\psi$ are known as polygamma functions [5, p. 16]:
\[
(5) \quad \psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1 - e^{-xt}}dt \quad (x > 0; n = 1, 2, \ldots).
\]

For more information, we refer to [1, p. 255] and the references given therein.

The main purpose of this paper is to establish the following conclusions.

**Theorem 1.** Let $m \geq 2$ be integer, $I = (0, +\infty)$ and let $F(x) = \frac{\Gamma(mx)}{x^{m-1}\Gamma^m(x)}$, $G(x) = \frac{\Gamma(mx)}{\Gamma^m(x)}$, $x \in I$. Then we have
\[
(1) \quad (-1)^n (\log F(x))^{(n)} > 0 \text{ for } x \in (0, \infty), n = 2, 3, \ldots.
\]
\[
(2) \quad \text{The function } \frac{1}{G(x)} \text{ is strictly logarithmically completely monotonic on } I.
\]

**Proof.** Using Gauss' multiplication formula of the gamma function [1, p. 256]
\[ (6) \quad \Gamma(mx) = (2\pi)^{\frac{1}{2}} (1-m) m^{mx - \frac{1}{2}} \frac{\Gamma(x + \frac{1}{m}) \Gamma(x + \frac{2}{m}) \cdots \Gamma(x + \frac{m-1}{m})}{\Gamma(x + 1)}. \]

and the recurrence formula

\[ (7) \quad \Gamma(x + 1) = x \Gamma(x), \]

we obtain

\[ (8) \quad F(x) = \frac{\Gamma(mx)}{x^{m-1} \Gamma^m(x)} = \frac{(2\pi)^{\frac{1}{2}} (1-m) m^{mx - \frac{1}{2}} \Gamma(x + \frac{1}{m}) \Gamma(x + \frac{2}{m}) \cdots \Gamma(x + \frac{m-1}{m})}{\Gamma(mx - 1)}. \]

Taking logarithm and differentiation, and using (5) yields

\[ (log F(x))' = m \log m + \psi(x + \frac{1}{m}) + \psi(x + \frac{2}{m}) + \cdots \]
\[ + \psi(x + \frac{m-1}{m}) - (m-1) \psi(x+1) \]
\[ = m \log m + \int_0^\infty (m-1) e^{-(x+1)t} e^{-x \frac{t}{m}} - \cdots - e^{-x \frac{m-1}{m} t} \frac{1}{1-e^{-t}} dt \]
\[ = m \log m + \int_0^\infty \sum_{i=1}^{m-1} \frac{e^{-t} - e^{-\frac{t}{m}}}{1-e^{-t}} e^{-xt} dt \]

and

\[ (log F(x))'' = - \int_0^\infty \sum_{i=1}^{m-1} \frac{e^{-t} - e^{-\frac{t}{m}}}{1-e^{-t}} t e^{-xt} dt > 0. \]

Therefore, we get

\[ (10) \quad (-1)^n (log F(x))^{(n)} = - \int_0^\infty \sum_{i=1}^{m-1} \frac{e^{-t} - e^{-\frac{t}{m}}}{1-e^{-t}} t^{n-1} e^{-xt} dt > 0 \]

for \( x > 0; \ n = 2, 3, \ldots \).

Next, from

\[ \frac{1}{G(x)} = (2\pi)^{\frac{1}{2}} (1-m) m^{mx - \frac{1}{2}} \frac{\Gamma^m(x)}{\Gamma^2(x) - \Gamma^m(x) x}, \]

we obtain, by taking logarithm and differentiation,

\[ (11) \quad \left( \log \frac{1}{G(x)} \right)' = -m \log m + \int_0^\infty \sum_{i=1}^{m-1} \frac{e^{-\frac{t}{m}} - 1}{1-e^{-t}} e^{-xt} dt < 0 \]

and

\[ (12) \quad (-1)^n \left( \log \frac{1}{G(x)} \right)^{(n)} = - \int_0^\infty \sum_{i=0}^{m-1} \frac{e^{-\frac{t}{m}} - 1}{1-e^{-t}} t^{n-1} e^{-xt} dt > 0 \]
for $x > 0$; $n = 2, 3, \ldots$.

Combining (11) and (12), we can easily draw the conclusion. The proof is complete.

\[ \square \]

**Corollary.** Let $m \geq 2$ be integer, $I = (-1/m, +\infty)$ and let $g(x) = \frac{\Gamma(mx + 1)}{\Gamma(mx)}$, $x \in I$. Then we have $(-1)^n(\log g(x))^{(n)} > 0$ for $x \in (-1/m, \infty)$, $n = 2, 3, \ldots$.

**Remark 1.** If we slightly modify the functions $F(x)$, $G(x)$ and $g(x)$, then we obtain several logarithmically completely monotonic functions, like, for example, $\frac{\Gamma(mx)}{m^{mx} \Gamma(mx)}$, $\frac{\Gamma(mx)}{m^{mx} \Gamma(mx) \Gamma(mx + 1)}$.

**Remark 2.** Theorem 1 and Corollary can be expressed in another way: the functions $(\log F(x))^n$ and $(\log g(x))^n$ are completely monotonic on $(0, \infty)$ and $(-1/m, +\infty)$ respectively.

Recall that [7, pp. 75–76] a function $f$ with $n$ arguments defined on $I^n$ is **Schur-convex** on $I^n$ if $f(x) \leq f(y)$ for each two $n$-tuples $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $I^n$ such that $x \prec y$ holds, where $I$ is an interval with nonempty interior. The relationship of majorization $x \prec y$ means that

\[ \sum_{i=1}^{k} x[i] \leq \sum_{i=1}^{k} y[i], \quad \sum_{i=1}^{n} x[i] = \sum_{i=1}^{n} y[i], \]

where $1 \leq k \leq n - 1$, $x[i]$ denotes the $i$th largest component in $x$.

It is well known that the function $x \mapsto \prod_{i=1}^{n} f(x_i), \ x \in I^n$ is **Schur-convex** (strictly **Schur-convex**) on $I^n$ if and only if $f$ is log-convex (strictly log convex) on $I$, where $f$ is a continuous nonnegative function defined on an interval $I \subset \mathbb{R}$.

Considering **Schur-convexity** of the function $\phi(x)$ mentioned in (1), we will extend it to general forms in the following theorems.

**Theorem 2.** Let $m \geq 2$ be integer. The function

\[ \phi(x) = \prod_{i=1}^{n} \frac{\Gamma(mx_i + 1)}{\Gamma^m(mx_i + 1)} \]

is strictly **Schur-convex** on $x = (x_1, \ldots, x_n) \in (-1/m, +\infty)^n$.

**Proof.** From Corollary we get that the function $g(x)$ is strictly log-convex on $(-1/m, \infty)$. Let $g(x_i) = \frac{\Gamma(mx_i + 1)}{\Gamma^m(x_i + 1)}$, $x_i \in (-1/m, \infty)$, $i = 1, 2, \ldots, n$. Hence $\phi(x) = \prod_{i=1}^{n} g(x_i)$ is strictly **Schur-convex** on $x = (x_1, \ldots, x_n) \in (-1/m, +\infty)^n$, where each $g(x_i)$ is strictly log-convex function.

\[ \square \]

**Acknowledgements.** The authors are indebted to the anonymous referee for many valuable comments and corrections in language expressions.
REFERENCES


School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China
E–mail: iaijun72@163.com

School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China
E–mail: weizhen@sina.com

School of Mathematics and Informatics, Research Institute of Applied Mathematics, Henan Polytechnic University, Jiaozuo City, Henan 454010, China
E–mail: chenchaoping@hpu.edu.cn, chenchaoping@sohu.com