THE NUMERICAL STABILITY OF A LAGUERRE-LIKE METHOD FOR THE SIMULTANEOUS INCLUSION OF POLYNOMIAL ZEROS

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Abstract The numerical stability of the fourth order iterative method of Laguerre’s type for the simultaneous inclusion of polynomial zeros is analyzed in the presence of rounding errors. We state conditions under which the convergence order of the considered method is preserved. If these conditions are relaxed, the convergence rate reduces to three.

1. INTRODUCTION

The problem of finding zeros of a polynomial is one of the oldest and the most important problems in the theory and practice. Iterative methods for the simultaneous determination of polynomial zeros, realized in interval arithmetic, produce resulting real or complex intervals containing the wanted zeros. Therefore, these methods can be regarded as a valid-selfated numerical tool giving the upper error bounds to approximations expressed by the semi-widths of the intervals or by the radii of the resulting disks. More details about simultaneous inclusion methods for polynomial zeros can be found, e.g., in [1], [8], [10] and references cited therein.

In this paper we investigate the fourth order iterative method of Laguerre’s type for the simultaneous inclusion of polynomial zeros, proposed in [9]. Basically we are interested in the following questions: Which conditions can guarantee the same convergence rate of the presented Laguerre-like iterative method in the...
The presence of round-off arising due to the use of floating point arithmetic of finite precision? Also, which order of convergence can be expected when the previous conditions are relaxed? The study of these questions is the main goal of this paper.

The construction of inclusion methods and their numerical stability analysis, presented in this paper, need the basic properties of the so-called circular complex arithmetic introduced by Gargantini and Henrici [5]. For the reader’s convenience we give some basic properties of circular complex arithmetic. More details can be found in the books [1], [8] and [10].

A disk $Z$ with center mid $Z = c$ and radius rad $Z = r$, that is $Z := \{ z : |z - c| \leq r \}$, will be denoted briefly by the parametric notation $Z = \{ c; r \}$. In circular complex arithmetic the following simple properties are valid:

\[
\begin{align*}
\{ c_1; r_1 \} \pm \{ c_2; r_2 \} &= \{ c_1 \pm c_2; r_1 + r_2 \}, \\
\alpha \{ c; r \} &= \{ \alpha c; |\alpha| r \} \quad (\alpha \in \mathbb{C}), \\
\{ c_1; r_1 \} \cdot \{ c_2; r_2 \} &= \{ c_1 c_2; |c_1| r_2 + |c_2| r_1 + r_1 r_2 \}, \\
\{ c_1; r_1 \} \cap \{ c_2; r_2 \} &= \emptyset \Leftrightarrow |c_1 - c_2| > r_1 + r_2, \\
|c_1 - c_2| \leq r_1 - r_2 &\Leftrightarrow \{ c_1; r_1 \} \subseteq \{ c_2; r_2 \}.
\end{align*}
\]

Consider now the inversion of a disk $Z = \{ c; r \}$ which does not contain the origin, that is, $|c| > r$ holds. Under the transformation $w(z) = 1/z$ this disk maps into the disk

\[
Z^{-1} = \left\{ \frac{1}{z} : z \in \{ c; r \} \right\} = \left\{ \frac{\bar{c}}{|c|^2 - r^2}; \frac{r}{|c|^2 - r^2} \right\}.
\]

Following the inversion, division is defined as

\[
Z_1 : Z_2 = Z_1 \cdot Z_2^{-1} \quad (0 \notin Z_2).
\]

It is easy to prove that, if $z \in Z$, then

\[
\max \{ 0, |\text{mid } Z - \text{rad } Z| \} \leq |z| \leq |\text{mid } Z| + |\text{rad } Z|.
\]

The square root of a disk $\{ c; r \}$ in the centered form, where $c = |c|e^{i\theta}$ and $|c| > r$, is defined as the union of two disjoint disks (see [2]):

\[
\{ c; r \}^{1/2} := \left\{ \sqrt{|c|}e^{i\theta/2}; \sqrt{|c|} - \sqrt{|c| - r} \right\} \bigcup \left\{ -\sqrt{|c|}e^{i\theta/2}; \sqrt{|c|} - \sqrt{|c| - r} \right\}.
\]

In what follows, disks in the complex plane will be denoted by capital letters.

### 2. Derivation of the Laguerre-like Interval Method

Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a monic polynomial with simple zeros $\zeta_1, \ldots, \zeta_n$ and let $\mathcal{I}_n := \{ 1, \ldots, n \}$ be the index set. For the point
$z = z_i \ (i \in \mathcal{I}_n)$ let us introduce

$$
\Sigma_{k,i} = \sum_{j \neq i}^{n} \frac{1}{(z_i - \zeta_j)^k}, \quad S_{k,i} = \sum_{j \neq i}^{n} \frac{1}{(z_i - z_j)^k} \quad (k = 1, 2),
$$

$$
q_i^* = n\Sigma_{2,i} - \frac{n}{n-1} \Sigma_{1,i}^2, \quad q_i = nS_{2,i} - \frac{n}{n-1} S_{1,i}^2,
$$

$$
\delta_{1,i} = \frac{P'(z_i)}{P(z_i)}, \quad \delta_{2,i} = \frac{P'(z_i)^2 - P(z_i)P''(z_i)}{P(z_i)^2}, \quad \varepsilon_i = z_i - \zeta_i.
$$

The following identity

$$
(5) \quad n \cdot -\delta_{1,i}^2 - q_i^* = \frac{1}{n-1} \left( \frac{n}{\varepsilon_i} - \delta_{1,i} \right)^2
$$

was proved in [9]. From (5) we obtain the fixed point relation

$$
(6) \quad \zeta_i = z_i - \frac{n}{\delta_{1,i} \pm \sqrt{(n-1)(n \cdot -\delta_{1,i}^2 - q_i^*)}} \quad (i \in \mathcal{I}_n).
$$

Let us assume that we have found nonintersecting disks $Z_1, \ldots, Z_n$ with the centers $z_i = \text{mid} Z_i$ and the radii $r_i = \text{rad} Z_i$ such that $\zeta_i \in Z_i \ (i \in \mathcal{I}_n)$. Taking the inclusion disks $Z_j$ instead of the zeros $\zeta_j$ in the expression for $q_i^*$, we obtain a circular extension $Q_i$ of $q_i^*$,

$$
(7) \quad Q_i = n \sum_{j \neq i}^{n} \left( \frac{1}{z_i - Z_j} \right)^2 - \frac{n}{n-1} \left( \sum_{j \neq i}^{n} \frac{1}{z_i - Z_j} \right)^2.
$$

In regard to the inclusion isotonicity and the definition of $q_i^*$ we have $q_i^* \in Q_i$. Furthermore, from the fixed point relation (6) we obtain

$$
(8) \quad \tilde{z}_i := z_i - \frac{n}{\delta_{1,i} + \left[ (n-1)(n \cdot -\delta_{1,i}^2 - Q_i) \right]^{1/2}} \quad (i \in \mathcal{I}_n).
$$

According to (4), the square root of a disk in (8) produces two disks; the symbol $\ast$ indicates that one of the two disks has to be chosen. That disk will be called a „proper“ disk. From (5) and the inclusion $q_i^* \in Q_i$ we conclude that proper disk is the one which contains $n/\varepsilon_i - \delta_{1,i}$. Taking into account (4), we have

$$
\left( (n-1)(n \cdot -\delta_{1,i}^2 - Q_i) \right)^{1/2} \ G_{1,i} \cup G_{2,i}, \quad \text{mid} G_{k,i} = g_{k,i}, \quad g_{1,i} = -g_{2,i}
$$

for $i \in \mathcal{I}_n, \ k = 1, 2$. The criterion for the choice of the proper disk is considered in [2] (see also [6]) and reads:

If the disks $Z_1, \ldots, Z_n$ are reasonably small, then we have to choose that disk (between $G_{1,i}$ and $G_{2,i}$) whose center minimizes $|P'(z_i)/P(z_i) - g_{k,i}| \ (k = 1, 2)$. 

If the denominator in (8) is not a disk containing 0, then \( \hat{Z}_i \) is a new outer circular approximation to the zero \( \zeta_i \), that is \( \zeta_i \in \hat{Z}_i \). Formula (8) suggests an algorithm of Laguerre’s type for the simultaneous inclusion of all simple zeros of a given polynomial \( P \):

\[
Z_i^{(m+1)} := z_i^{(m)} - \frac{n}{\delta_{1,i}^{(m)} + \left[ (n-1)(n\delta_{2,i}^{(m)} - (\delta_{1,i}^{(m)})^2 - Q_1^{(m)}) \right]^{1/2}},
\]

where \( i \in \mathcal{I}_n; m = 0, 1, \ldots \) and \( z_i^{(m)} = \text{mid} Z_i^{(m)} \). The interval method (9) was recently stated in \([9]\); it was proved there that its \( R \)-order of convergence is four.

**3. ITERATIVE METHOD IN THE PRESENCE OF ROUNDING ERRORS**

Any iterative procedure for finding the zeros of a function \( f \) has to terminate in a finite number of iterations in the presence of round-off (see \([3]\) for details). The termination will happen as soon as the absolute value of the rounding error involved in the evaluation of \( f \) in the proximity of 0 is of the same order of magnitude as \( |f| \).

Let \( P, P', P'' \) denote the given polynomial and its derivatives, and let \( \Delta P, \Delta P', \Delta P'' \) be the upper bounds for the absolute value of the round-off appeared in the evaluation of \( P, P', P'' \), respectively. These errors can be evaluated according to the rounding-error analysis introduced by Wilkinson \([12]\) and depend on the number of significant digits of the mantissa for floating-point arithmetic operations.

The application of the iterative formula (9) requires that the single errors involved in the evaluation of \( P, P', P'' \) are included in the determination of \( \delta_{1,i} \) and \( \delta_{2,i} \) because of the presence of rounding errors. Accordingly, we have to replace \( P \) by the disk \( T_0 = \{ P; \Delta P \} \) and the derivatives \( P^{(i)} \) by the disks \( T_i = \{ P^{(i)}; \Delta P^{(i)} \} (i = 1, 2) \) with the centers \( P, P', P'' \) and the radii \( \Delta P, \Delta P', \Delta P'' \). Circular extension of the scalar quantity \( \delta_{k,i} \), obtained by the above substitutions, will be denoted by \( D_{k,i} \) \((k = 1, 2)\). According to the properties of circular arithmetic, it follows that \( \delta_{k,i} \in D_{k,i} \) \((k = 1, 2)\).

Since in circular arithmetic the inversion of a disk is an exact operation, while the multiplication is not, in order to obtain the disks \( D_{k,i} \) as small as possible, before the substitution of \( P, P', P'' \) by the corresponding disks \( T_0, T_1, T_2 \) it is advisable to perform cancellations and single division of the addends in the numerator of the developed expression of \( \delta_{2,i} \) by the denominator \( P^2 \). In this way we obtain

\[
\delta_{1,i} = \frac{P'(z_i)}{P(z_i)}, \quad D_{1,i} = \frac{T_1}{T_0},
\]

\[
\delta_{2,i} = \frac{P'(z_i)^2 - P(z_i)P''(z_i)}{P(z_i)^2}, \quad D_{2,i} = \left( \frac{T_1}{T_0} \right)^2 - \frac{T_2}{T_0},
\]

with

\[
\text{rad } D_{2,i} < \text{rad } \frac{T_1^2 - T_2 T_0}{T_0^2}.
\]
In the sequel, we shall operate with total rounding errors $\phi_{k,i} = \text{rad} D_{k,i}$, which incorporate the single errors concerning the evaluation of $P, P', P''$.  

In order to provide that the inversion of the disk $T_0 = \{P; \Delta P\}$ (which appears in the evaluation of $T_k$) also produces a disk, we shall require $\Delta P < |P|$ in our analysis of the numerical stability of the algorithm (9). Also, as it was already said, the stopping criterion which determines the maximum number of iterations, is based on the comparison of $\Delta P^{(m)}_i$ with $|\Delta P(z_i^{(m)})|$ for each $i$.  

In this paper we shall use the following notation. At the $m$-th iteration we denote the radius of $D_{k,i}^{(m)} (k = 1, 2)$ by $\phi_{k,i}^{(m)}$. Also, we introduce the abbreviations  

$$
\phi_k^{(m)} = \max_{1 \leq i \leq n} \phi_{k,i}^{(m)}, \quad r_k^{(m)} = \max_{1 \leq i \leq n} r_{k,i}^{(m)}, \quad \rho_k^{(m)} = \min_{1 \leq i,j \leq n} \left\{ |z_i^{(m)} - z_j^{(m)}| - r_{j,k}^{(m)} \right\}.
$$

We note that only the scalars $\delta_{k,i}(z_i^{(m)})$ are replaced by the disks $D_{k,i}(z_i^{(m)})$, $i \in I_n, m = 0, 1, \ldots$. Thus, the algorithm (9) for the simultaneous inclusion of all simple zeros of a given polynomial in the presence of rounding errors has the form  

$$
Z_i^{(m+1)} := z_i^{(m)} - \frac{n}{D_{1,i}^{(m)} + [(n-1)(nD_{2,i}^{(m)} - (D_{1,i}^{(m)})^2 - Q_i^{(m)})]^{1/2}},
$$

for $i \in I_n$ and $m = 0, 1, \ldots$.  

4. CONVERGENCE STABILITY: THE CASE $\phi_k < r^2/\rho^{k+2}$  

In what follows we shall investigate the dependence of the convergence order of the method (10) when the absolute value of the round-off in the evaluation of the polynomial in the proximity of 0 is small compared to the absolute value of the polynomial itself, i.e., when $\phi_k < r^2/\rho^{k+2}$ $(k = 1, 2)$. In the analysis we will assume that $\Delta P^{(m)}_i < |P(z_i^{(m)})|$.  

For simplicity, we shall omit the iteration index always when there is no possibility of misunderstanding.  

Before the convergence analysis we will prove two necessary lemmas assuming that $n \geq 3$.  

**Lemma 1.** Let $\eta = 5n(n-1)r/\rho^3$ and let the inequality  

$$
\rho > 3(n-1)r
$$

hold. Then  

$$
Q_i \subset \{q_i; \eta\}.
$$

**Proof.** Using inclusion derived in [6]  

$$
\left( \frac{1}{z_i - Z_j} \right)^k \subset \left\{ \frac{1}{(z_i - z_j)^k}; \frac{kr}{\rho^{k+1}} \right\}
$$
and the inequalities

\[
\frac{1}{|z_i - \zeta_j|} \leq \frac{1}{\rho} \quad \text{and} \quad \frac{1}{|z_i - z_j|} \leq \frac{1}{\rho}
\]

(since \(|z_i - \zeta_j| \geq |z_i - z_j| - |z_j - \zeta_j| \geq |z_i - z_j| - r_j \geq \rho\)), we obtain

\[
Q_i \subset n \sum_{j \neq i} \left\{ \frac{1}{(z_i - z_j)^2}; \frac{2r}{\rho^2} \right\} - \frac{n}{n-1} \left\{ \frac{1}{(z_i - z_j)^2}; \frac{r}{\rho^2} \right\}^2
\]

Let us introduce the following abbreviations:

\[
Y_i = nD_{2,i} - D_{1,i}^2 - Q_i, \quad y_i = n \text{mid} D_{2,i} - (\text{mid} D_{1,i})^2 - q_i, \quad \hat{\eta} = \eta + n\phi_2 + \phi_1^2, \quad v_i = \frac{(n-1)(q_i^* - q_i)}{(n/\varepsilon_i - \delta_{1,i})^2}, \quad d = 4.6n(n-1)^2r^2/\rho^3.
\]

**Lemma 2.** Let \(\Delta P_i < |P(z_i)|, \phi_k \leq r^2/\rho^{k+2} (k = 1, 2)\) and let the inequality (11) hold. Then

\(\)

(i) \(|y_i| > \frac{1}{r^2} \left( n - \frac{41}{18} \right)\);

(ii) \(\hat{\eta} < 5.2n(n-1)\frac{r}{\rho^2}\);

(iii) \(|y_i| - \hat{\eta} > \frac{1}{r^2} \left( n - \frac{121}{50} \right)\);

(iv) \(\sqrt{(n-1)}[y_i; \hat{\eta}] \subset \left\{ \sqrt{(n-1)}y_i; d \right\}\);

(v) \(\sqrt{1 + v_i} \in \left\{ 1; \frac{3}{10} \frac{\varepsilon_i}{\rho} \right\}\).

**Proof.** Of (i): Using the identities

\[
\delta_{1,i} = \frac{P'(z_i)}{P(z_i)} = \sum_{j=1}^{n} \frac{1}{z_i - \zeta_j}
\]
and

$$\delta_{2,i} = \frac{P'(z_i)^2 - P(z_i)P''(z_i)}{P(z_i)^2} = \sum_{j=1}^{n} \frac{1}{(z_i - \zeta_j)^2}$$

we find

$$y_i = n\left(\frac{1}{\varepsilon_i} + \Sigma_{2,i}\right) - \left(\frac{1}{\varepsilon_i} + \Sigma_{1,i}\right)^2 - n\left(S_{2,i} - \frac{1}{n-1}S_{1,i}^2\right)$$

$$= \frac{n - 1 - 2\varepsilon_i\Sigma_{1,i}}{\varepsilon_i^2} + n\Sigma_{2,i} - \Sigma_{1,i}^2 - nS_{2,i} + \frac{n}{n-1}S_{1,i}^2.$$ 

Starting from this expression and (11), we find

$$|\Sigma_{k,i}| \leq \frac{n - 1}{\rho^k} \quad \text{and} \quad |S_{k,i}| \leq \frac{n - 1}{\rho^k}$$

for \(k = 1, 2\). Now we have

$$|y_i| \geq \left|\frac{n - 1 - 2\varepsilon_i\Sigma_{1,i}}{\varepsilon_i^2} - n|\Sigma_{2,i}| - |\Sigma_{1,i}|^2 - n|S_{2,i}| - \frac{n}{n-1}|S_{1,i}|^2\right.$$ 

$$\geq \frac{n - 1 - 2r \frac{n - 1}{\rho}}{r^2} - \frac{3n(n-1)}{\rho^2} - \frac{(n-1)^2}{\rho^2}$$

$$> \frac{n - 5/3}{r^2} - \frac{(n-1)(4n-1)}{\rho^2} > \frac{1}{r^2} \left(n - \frac{5}{3} - \frac{4n-1}{9(n-1)}\right) \geq \frac{1}{r^2} \left(n - \frac{41}{18}\right).$$

Of (ii): Using inequalities $\phi_1^2 < r^2/\rho^4$, $\phi_2 < r^2/\rho^4$ and Lemma 1, we estimate

$$\hat{\eta} = \eta + n\phi_2 + \phi_2^2 < 5n(n-1) \frac{r}{\rho^4} + (n+1) \frac{1}{3(n-1)} \frac{r}{\rho^4}$$

$$= \frac{15n^3 - 30n^2 + 16n + 1}{3(n-1)} \frac{r}{\rho^4} < 5.2n(n-1) \frac{r}{\rho^4}.$$ 

Of (iii): Using (i) and (ii) we obtain

$$|y_i| - \hat{\eta} > \frac{1}{r^2} \left(n - \frac{41}{18}\right) - \frac{5n(n-1)r}{\rho^3} - \frac{(n+1)r^2}{\rho^4}$$

$$> \frac{1}{r^2} \left(n - \frac{29}{12} - (n+1) \frac{1}{(3(n-1))^2}\right)$$

$$> \frac{1}{r^2} \left(n - \frac{1045}{432}\right) > \frac{1}{r^2} \left(n - \frac{121}{50}\right) > 0.$$ 

Of (iv): According to (iii) we conclude that $|y_i| > \hat{\eta}$ so that we can apply (4) in order to obtain the square root of a disk $\{y_i; \hat{\eta}\}$,

$$\sqrt{\{y_i; \hat{\eta}\}} = \left\{ y_{i}^{1/2}; \sqrt{|y_i|} - \sqrt{|y_i| - \hat{\eta}} \right\} = \left\{ y_{i}^{1/2}; R_i \right\},$$
where
\[ R_i = \frac{\hat{\eta}}{\sqrt{|y_i|} + \sqrt{|y_i| - \hat{\eta}}}. \]

Using (i), (ii) and (iii) we now estimate
\[
\sqrt{n-1} R_i \leq 5.2n(n-1) \frac{\sqrt{n-1}}{\sqrt{n-41} + \sqrt{n-121}} \frac{r^2}{\rho^3} < 4.6 n(n-1) \frac{r^2}{\rho^3},
\]
since
\[
\frac{\sqrt{n-1}}{\sqrt{n-41} + \sqrt{n-121}} < 0.88
\]
for all \( n \geq 3 \). We finally conclude that
\[
\sqrt{(n-1)[y_i; \hat{\eta}]} = \{ \sqrt{(n-1)y_i}; \sqrt{n-1}R_i \} \subset \{ \sqrt{(n-1)y_i}; 4.6n(n-1) \frac{r^2}{\rho^3} \}.
\]

Of (v): Using the expressions for \( q_i \) and \( q_i^* \) we find
\[
q_i^* - q_i = -n \sum_{j \neq i} \frac{\varepsilon_j}{(z_i - \zeta_j)(z_i - z_j)} \left( \frac{1}{z_i - \zeta_j} + \frac{1}{z_i - z_j} \right)
+ \frac{n}{n-1} \sum_{j \neq i} \frac{\varepsilon_j}{(z_i - \zeta_j)(z_i - z_j)} \left( \sum_{j \neq i} \frac{1}{z_i - \zeta_j} + \sum_{j \neq i} \frac{1}{z_i - z_j} \right).
\]

By the inequalities (14), \( |\varepsilon_i| \leq r \) and \( |z_i - z_j| \geq \rho \), we estimate
\[
|q_i^* - q_i| \leq \frac{2n(n-1)r}{\rho^3} + \frac{n}{n-1} \frac{2(n-1)^2r}{\rho^3} = \frac{4n(n-1)r}{\rho^3}.
\]

Since
\[
|\text{mid } D_{k,i} - \delta_{k,i}| \leq \phi_k,
\]
we find
\[
|n - \varepsilon_i \text{ mid } D_{1,i}| \geq |n - \varepsilon_i (\delta_{1,i} + \phi_1)| \geq |n - \varepsilon_i \delta_{1,i}| - |\varepsilon_i \phi_1|
\geq n - 1 - |\varepsilon_i| \sum_{j \neq i} \frac{1}{|z_i - \zeta_j|} - \frac{1}{(3(n-1))^3}
> \frac{5(n-1)}{6} - \frac{1}{27(n-1)^3}
= \frac{45n^4 - 180n^3 + 270n^2 - 180n + 43}{54(n-1)^3} > \frac{4}{5} (n-1)
\]
and

\[
|n - \epsilon_i \text{ mid } D_{1,i}| \leq |n - \epsilon_i (\delta_{1,i} - \phi_1)| \leq |n - \epsilon_i \delta_{1,i}| + |\epsilon_i \phi_1| \\
\leq n - 1 + |\epsilon_i| \sum_{j \neq i} |\epsilon_i - \zeta_j| + \frac{1}{(3(n - 1))^3} \\
< \frac{7(n - 1)}{6} + \frac{1}{27(n - 1)^3} \\
= \frac{63n^4 - 252n^3 + 378n^2 - 252n + 65}{54(n - 1)^3} < \frac{6}{5}(n - 1).
\]

Using (15) and (16) we obtain

\[
|v_i| \leq \left( n - 1 \right)|q_i^* - q_i| < \frac{(n - 1) \cdot 4n(n - 1)|\epsilon_i|^2}{\rho^2} < \frac{16}{25} (n - 1)^2 \\
\leq \frac{25n}{4} \cdot \frac{r^2}{\rho^2} \cdot |\epsilon_i| < \frac{53}{100} \rho < \frac{14 |\epsilon_i|}{25 \rho} < \frac{14}{25} |\epsilon_i| =: \gamma_i.
\]

Let \( V_i := \{0; \gamma_i\} \) be a disk, then \( v_i \in V_i \) and by (4) we find

\[
\sqrt{1 + v_i} \in \sqrt{1 + V_i} = \sqrt{1: \gamma_i} = \left\{ 1; \frac{\gamma_i}{1 + \sqrt{1 - \gamma_i}} \right\}.
\]

Since

\[
\gamma_i = \frac{14 |\epsilon_i|}{25 \rho} < \frac{14 r}{25 \rho} < \frac{7}{75},
\]

then

\[
\frac{1}{1 + \sqrt{1 - \gamma_i}} < \frac{13}{25},
\]

so that

\[
\sqrt{1 + v_i} \in \left\{ 1; \frac{14}{25} \cdot \frac{13}{25} \cdot \frac{|\epsilon_i|}{\rho} \right\} \subset \left\{ 1; \frac{3}{10} \cdot \frac{|\epsilon_i|}{\rho} \right\}.
\]

Using Lemmas 1 and 2 we are now able to prove that the order of convergence of the method (10) is four under specific initial conditions.

**Theorem 1.** Let the interval sequences \( \{z_i^{(m)}\} \) \( (i = 1, \ldots, n) \) be defined by the iterative formula (10). If \( \Delta I_i^{(m)} \leq |P(z_i^{(m)})|, z_i^{(m)} \leq (r^{(m)})^2 / (\rho^{(m)})^{k+2} \ (k = 1, 2) \) and

\[
\rho^{(0)} > 3(n - 1)r^{(0)}
\]

hold, then for each \( i = 1, \ldots, n \) and \( m = 0, 1, \ldots \) we have

\[
1^o \ \zeta_i \in Z_i^{(m)};
\]
\[2^{\circ} \quad r^{(m+1)} < \frac{5.2(n-1)(r^{(m)})^4}{(\rho^{(0)} - \frac{5}{4} r^{(0)})^3}.
\]

**Proof.** We will prove the assertion \(1^\circ\) by induction. Assume that \(\zeta_i \in Z_i^{(m)}\) for \(i \in I_n\) and \(m \geq 1\). Then

\[
\frac{n}{\sum_{j \neq i} (z_i^{(m)} - \zeta_j)^2} - \frac{n}{n-1} \left(\sum_{j \neq i} z_i^{(m)} - \zeta_j\right)^2 \in \mathbb{Q}^{(m)}
\]

and, on the basis of (6) and the inclusion isotonicity, it follows

\[
\zeta_i \in z_i^{(m)} - \frac{n}{D^{(m)}_{i,i} + \left[\left(n-1\right)\left(nD_{2,i}^{(m)} - D_{1,i}^{(m)}\right)^2 - Q_i^{(m)}\right]^{1/2}} = Z_i^{(m+1)}.
\]

Since \(\zeta_i \in Z_i^{(0)}\), we obtain by induction that \(\zeta_i \in Z_i^{(m)}\) for each \(m = 0, 1, \ldots\)

Let us prove now that the interval method (10) has the order of convergence equal to four (assertion \(2^\circ\)) under the conditions of Theorem 1. We will use induction and start with \(m = 0\). For simplicity, all indices are omitted and all quantities in the first iteration are denoted by \(\hat{\cdot}\).

Using the inclusion (12), the assertion (iv) of Lemma 2 and circular arithmetic operations, from the iterative formula (10) we obtain

\[
\hat{Z}_i \subset z_i - \frac{n}{D_{1,i}^{(m)} + \left\{\sqrt{\left(n-1\right)y_i; d}\right\}} = z_i - \frac{n}{\left\{u_i; d + \phi_1\right\}}.
\]

where we put

\[
u_i = \text{mid} D_{1,i} + \left\{\left(n-1\right)y_i\right\}^{1/2}.
\]

Using the identity (5) we find

\[
y_i = n \text{mid} D_{2,i} - q_i = \frac{1}{n-1} \left(n/\varepsilon_i - \text{mid} D_{1,i}\right)^2 + q_i - q_i = \frac{1}{n-1} (n/\varepsilon_i - \text{mid} D_{1,i})^2 (1 + v_i).
\]

According to this and the assertion (v) of Lemma 2 we obtain

\[
u_i = \text{mid} D_{1,i} + \left\{\left(n-1\right)y_i\right\}^{1/2} = \text{mid} D_{1,i} + (n/\varepsilon_i - \text{mid} D_{1,i}) \sqrt{1 + v_i},
\]

\[
u_i \in \text{mid} D_{1,i} + (n/\varepsilon_i - \text{mid} D_{1,i}) \left\{1; \frac{3}{10} \frac{\varepsilon_i}{\rho} \right\} = \frac{n}{\varepsilon_i} \left\{ \frac{3}{10} n - \varepsilon_i \text{mid} D_{1,i} \right\} =: U_i.
\]

Here we have taken into account the criterion for the selection of the proper value of the square root (see Section 2). Using (3) and (17) we find

\[
u_i > |\text{mid} U_i| - \text{rad} U_i = \frac{n}{\varepsilon_i} \frac{3}{10} \frac{|n - \varepsilon_i \text{mid} D_{1,i}|}{\rho} > \frac{n}{r} \frac{6(n-1)}{5} \frac{3}{10} > \frac{1}{r} \left(\frac{n}{r - \frac{3}{25}}\right).
\]
Using (11) and (20) we estimate
\[ |u_i| - (d + \phi_1) > \frac{1}{r} \left( n - \frac{3}{25} \right) - \left( 4.6n(n - 1) \frac{r^2}{\rho^3} + \frac{r^2}{\rho^3} \right) > \frac{1}{r} (n - 0.26) > 0. \]

According to the last inequality we see that 0 \( \notin \{u_i; d + \phi_1\} \) so that iterative process (10) is well defined and \( \hat{Z}_i \) is a closed disk. From (19) we find
\[ \hat{Z}_i \subset z_i - \frac{n}{|u_i|^2 - (d + \phi_1)^2} \{ \frac{d + \phi_1}{|u_i|^2 - (d + \phi_1)^2} \}, \]
whence
\[ \hat{r}_i = \text{rad} \hat{Z}_i < \frac{n(d + \phi_1)}{|u_i|^2 - (d + \phi_1)^2}. \]

Using the lower bound of \(|u_i|\) given by (20), from (21) we obtain
\[ \hat{r} < 5.2(n - 1) \frac{r^4}{\rho^3}, \]
and
\[ \hat{r} < 5.2(n - 1)r \frac{r^3}{\rho^3} < \frac{r}{20}. \]

According to a geometric construction and the fact that the disks \( Z_i^{(m)} \) and \( Z_{i+1}^{(m+1)} \) must have at least one common point (the zero \( \zeta_i \)), the following relation can be derived (see [3])
\[ \rho^{(m+1)} \geq \rho^{(m)} - \rho^{(m)} - 3r^{(m+1)}. \]

Using the inequalities (23) and (24) (for \( m = 0 \)), we obtain
\[ \rho^{(1)} \geq \rho^{(0)} - 3r^{(1)} > 3(n - 1)r^{(0)} - 3 \frac{r^{(0)}}{20} > 20r^{(1)}(3n - 1) - 3/20, \]
wherefrom
\[ \rho^{(1)} > 3(n - 1)r^{(1)}. \]

This is the condition (11) for the index \( m = 1 \), which means that all assertions of Lemmas 1 and 2 hold for \( m = 1 \).

Using the definition of \( \rho \) and (25), for arbitrary pair of indices \( i, j \) (\( i \neq j \)) we have
\[ |z_i^{(1)} - z_j^{(1)}| \geq \rho^{(1)} > 3(n - 1)r^{(1)} > 2r^{(1)} > r_i^{(1)} + r_j^{(1)}. \]

From (26) we conclude (according to (1)) that disks \( Z_i^{(1)}, \ldots, Z_n^{(1)} \) produced by (10) are disjoint.
Applying induction with the argumentation used for the derivation of (22), (23), (25) and (26) we prove that for each $m = 0, 1, \ldots$, the disks $Z_i^{(m)}, \ldots, Z_n^{(m)}$ are disjoint and the following relations are true:

\begin{align*}
(27) & \quad r^{(m+1)} < \frac{5.2(n - 1)(r^{(m)})^4}{(\rho^{(m)})^3}, \\
(28) & \quad r^{(m+1)} < \frac{r^{(m)}}{20}, \\
(29) & \quad \rho^{(m)} > 3(n - 1)r^{(m)}.
\end{align*}

In addition we note that the last inequality (29) means that the assertions of Lemmas 1 and 2 are true for each $m = 0, 1, \ldots$.

By the successive application of (24) and (28) we obtain

$$\rho^{(m)} > \rho^{(0)} - \frac{5}{4} r^{(0)}.$$ 

According to the last inequality and (27) we obtain

$$r^{(m+1)} < \frac{5.2(n - 1)(r^{(m)})^4}{(\rho^{(0)} - \frac{5}{4} r^{(0)})^3},$$

which means that the order of convergence of the iterative method (10) in the presence of rounding errors remains four under the conditions given in Theorem 1.

\[\square\]

5. CONVERGENCE ANALYSIS: THE CASE $\phi_k < r/\rho^{k+1}$

Here we shall investigate the convergence order of the method (10) when the absolute value of the round-off in the evaluation of the polynomial in the proximity of 0 is $\phi_k < r/\rho^{k+1} (k = 1, 2)$. As in the previous section, in our analysis we will assume that $\Delta P_i^{(m)} < |P(z_i^{(m)})|$.

As in Section 4, we will first give some inclusions, necessary for the proof of the convergence theorem.

Let us introduce $d = 5.2n(n - 1)\frac{r^2}{\rho^2}$.

**Lemma 3.** Let $\Delta P_i < |P(z_i)|$, $\phi_k \leq r/\rho^{k+1} (k = 1, 2)$ and let the inequality (11) hold. Then

\begin{align*}
(i) & \quad \sqrt{(n - 1)}\{y_i; \hat{\eta}\} \subset \{\sqrt{(n - 1)}y_i; d\}; \\
(ii) & \quad \sqrt{1 + v_i} \in \left\{1; \frac{3}{10} \cdot \frac{|\varepsilon_i|}{\rho}\right\}.
\end{align*}
The numerical stability of a Laguerre-like method

The proof of this lemma will be omitted since it is similar to that of Lemma 2.

Using Lemmas 1 and 3 we are now able to prove that the order of convergence of the method (10) is three under the relaxed initial conditions.

**Theorem 2.** Let the interval sequences \( \{ Z_i^{(m)} \} \) \((i = 1, \ldots, n)\) be defined by the iterative formula (10). If \( \Delta r_i^{(m)} < |P(z_i^{(m)})|, \phi_k^{(m)} \leq \left( r^{(m)} \right)^{k+1} \) \((k = 1, 2)\) and

\[
\rho(0) > 3(n-1)r^{(0)},
\]

then for each \( i = 1, \ldots, n \) and \( m = 0, 1, \ldots \) we have

1° \( \zeta_i \in Z_i^{(m)}; \)

2° \( r^{(m+1)} < \frac{1.2(n-1)(r^{(m)})^3}{\left( \rho(0) - \frac{4}{3}r^{(0)} \right)^2}. \)

**Proof.** The proof of the assertion 1° is the same as in Theorem 1.

Let us prove now that the interval method (10) has the order of convergence equal to three (assertion 2°). Using the inclusion (12), the assertion (i) of Lemma 3 and circular arithmetic operations, from the iterative formula (10) we obtain

\[
\hat{Z}_i \subset z_i - \frac{n}{D_{1,i} + \{ \sqrt{(n-1)y_i}; d \}^*} = z_i - \frac{n}{\{ u_i; d + \phi_1 \}},
\]

where we put

\[
u_i = \frac{1}{n-1} \left( n/\varepsilon_i - \text{mid} D_{1,i} \right)^2 + q_i^* - q_i = \frac{1}{n-1} \left( n/\varepsilon_i - \text{mid} D_{1,i} \right)^2 (1 + \nu_i).
\]

According to this and the assertion (ii) of Lemma 3 we obtain

\[
u_i = \text{mid} D_{1,i} + [(n-1)y_i]^{1/2} = \text{mid} D_{1,i} + (n/\varepsilon_i - \text{mid} D_{1,i}) \sqrt{1 + \nu_i},
\]

\[
u_i = \text{mid} D_{1,i} + [(n-1)y_i]^{1/2} = \text{mid} D_{1,i} + (n/\varepsilon_i - \text{mid} D_{1,i}) \sqrt{T + \nu_i},
\]

where we put

\[
u_i = \text{mid} D_{1,i} + [(n-1)y_i]^{1/2} = \frac{n}{10} \frac{3 |\varepsilon_i|}{\rho} = \left\{ \frac{n}{10} \frac{3 |\varepsilon_i|}{\rho}, \frac{n - \varepsilon_i \text{mid} D_{1,i}}{\rho} \right\} =: U_i.
\]

Using (3) and the estimation

\[
|n - \varepsilon_i \text{mid} D_{1,i}| \leq |n - \varepsilon_i (\delta_{1,i} - \phi_1)| \leq |n - \varepsilon_i \delta_{1,i} + |\varepsilon_i \phi_1|
\]

\[
\leq n - 1 + |\varepsilon_i| \sum_{j \neq i} \frac{1}{|\varepsilon_i - \zeta_j|} + \frac{1}{(3(n-1))^2} \leq \frac{7(n-1)}{6} + \frac{1}{9(n-1)^2}
\]

\[
= \frac{21n^3 - 63n^2 + 63n - 19}{18(n-1)^2} < \frac{6}{5} (n-1),
\]
we find

\[ |u_i| > |\text{mid } U_i| - \text{rad } U_i = \frac{n}{|\varepsilon_i|} - \frac{3}{10} \frac{|n - \varepsilon_i| \text{mid } D_{1,i}|}{\rho} \]

\[ > \frac{n}{r} \left( \frac{6(n - 1)}{5} \frac{3}{10} \frac{1}{\rho} \right) = \frac{1}{r} \left( n - \frac{3}{25} \right). \]

Following (11) and (31) we estimate

\[ |u_i| - (d + \phi_1) > \frac{1}{r} \left( n - \frac{3}{25} \right) - \left( 5.2 n (n - 1) \frac{r^2}{\rho^2} + \frac{r}{\rho^2} \right) > \frac{1}{r} (n - 0.27) > 0. \]

According to the last inequality we see that 0 \( \notin \{ u_i; d + \phi_1 \} \) so that iterative process (10) is well defined and \( \hat{Z}_i \) is a closed disk. From (19) we find

\[ \hat{Z}_i \subset z_i - n \left\{ \frac{\bar{u}_i}{|u_i|^2 - (d + \phi_1)^2};\frac{d + \phi_1}{|u_i|^2 - (d + \phi_1)^2} \right\}, \]

whence

\[ \hat{r}_i = \text{rad } \hat{Z}_i < \frac{n (d + \phi_1)}{|u_i|^2 - (d + \phi_1)^2}. \]

Using the lower bound of \( |u_i| \) given by (31), from (32) we obtain

\[ \hat{r} < 1.2 (n - 1) \frac{r^3}{\rho^2} \]

and

\[ \hat{r} < 1.2 (n - 1) \frac{r^2}{\rho^2} < 7 \frac{r}{100}. \]

Starting from the inequality (24) and using the inequality (34) (for \( m = 0 \)), we obtain

\[ \rho^{(1)} \geq \rho^{(0)} - 3r^{(1)} > 3(n - 1) r^{(0)} - 3 \frac{7r^{(0)}}{100} > \frac{100}{7} r^{(1)} (3(n - 1) - 1 - 21/100), \]

wherefrom

\[ \rho^{(1)} > 3(n - 1) r^{(1)}. \]

This is the condition (11) for the index \( m = 1 \), which means that all assertions of Lemmas 1 and 3 hold for \( m = 1 \).

Using the definition of \( \rho \) and (35), for arbitrary pair of indices \( i, j \) (\( i \neq j \)) we have

\[ |z_i^{(1)} - z_j^{(1)}| \geq \rho^{(1)} > 3(n - 1) r^{(1)} > 2r^{(1)} \geq r_i^{(1)} + r_j^{(1)}. \]
From (36) we conclude (according to (1)) that disks $Z_1^{(1)}, \ldots, Z_n^{(1)}$ produced by (10) are nonintersecting.

Applying induction with the argumentation used for the derivation of (33), (34), (35) and (36) we prove that for each $m = 0, 1, \ldots$, the disks $Z_1^{(m)}, \ldots, Z_n^{(m)}$ are disjoint and the following relations are true:

\begin{align*}
(37) & \quad r^{(m+1)} < \frac{1.2(n-1)(r^{(m)})^3}{(\rho^{(m)})^2}, \\
(38) & \quad r^{(m+1)} < \frac{7r^{(m)}}{100}, \\
(39) & \quad \rho^{(m)} > 3(n-1)r^{(m)}.
\end{align*}

The last inequality (39) means that the assertions of Lemmas 1 and 3 are true for each $m = 0, 1, \ldots$.

By the successive application of (24) and (38) we obtain

$$\rho^{(m)} > \rho^{(0)} - \frac{4}{3}r^{(0)}.$$  

According to the last inequality and (37) we obtain

$$r^{(m+1)} < \frac{1.2(n-1)(r^{(m)})^3}{(\rho^{(0)} - \frac{4}{3}r^{(0)})^2}.$$  

Therefore, under the relaxed conditions given in Theorem 2, the order of convergence of the iterative method (10) reduces to three.

6. CONVERGENCE ANALYSIS: THE CASE $\phi_k \leq \min(1, 1/\rho^k)$

In this case the influence of the rounding error, which has the power order as a constant is sufficient to decrease the convergence order of the method (10) to two. Here, we will distinguish two cases, $\rho < 1 \Rightarrow \min(1, 1/\rho^k) = 1$ and $\rho > 1 \Rightarrow \min(1, 1/\rho^k) = 1/\rho^k$. In both cases $1/\rho^k > \min(1, 1/\rho^k) (\geq \phi_k)$. Using this bound we conclude that $\phi_k^2 < 1/\rho^2$. As before, we will assume that $\Delta P_i^{(m)} < |P(z_i^{(m)})|$. Let $d = 1.7n(n-1)\frac{r}{\rho^2}$.

Lemma 4. Let $\Delta P_i < |P(z_i)|$, $\phi_k \leq \min(1, 1/\rho)$ and let the inequality (11) hold. Then

\begin{itemize}
  \item[(i)] $\sqrt{(n-1)} [y_i; \tilde{\eta}] \subset \{\sqrt{(n-1)}y_i; d\}$;
  \item[(ii)] $\sqrt{1 + v_i} \in \left\{1, \frac{8}{25} \cdot \frac{|\varepsilon_i|}{\rho}\right\}$.
\end{itemize}
The proof of this lemma is similar to that of Lemma 2.

Using Lemmas 1 and 4 we are now able to prove that the order of convergence of the method (10) is two, which is the subject of the following theorem.

**Theorem 3.** Let the interval sequences \( \{Z_i^{(m)}\} \) \((i = 1, \ldots, n)\) be defined by the iterative formula (10). If \( \Delta P_i^{(m)} < |P(z_i)^{(m)}|, \phi_k^{(m)} \leq 1/(\rho^{(m)})^k \) \((k = 1, 2)\) and

\[ \rho^{(0)} > 3(n - 1)r^{(0)}, \]

then for each \( i = 1, \ldots, n \) and \( m = 0, 1, \ldots \) we have

1. \( \zeta_i \in Z_i^{(m)}; \)

2. \( r^{(m+1)} < \frac{13(n - 1)(r^{(m)})^2}{25(\rho^{(0)} - 2r^{(0)})}. \)

The proof of this theorem is based on Lemmas 1 and 4 and goes in the similar way as the proof of Theorems 1 and 2. For this reason, it will be omitted.

**Remark.** In practice, the case \( \phi_k \leq \min(1, 1/\rho^k) \), meaning that the rounding errors have the order as a constant (that is, \( \phi_k = O(1) \)), cannot appear since the precision of the employed computer arithmetic is much higher.

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The numerical stability of a Laguerre-like method


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