ON A SECOND-ORDER STEP-SIZE ALGORITHM

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Abstract: In this paper we present a modification of the second-order step-size algorithm. This modification is based on the so called "forcing functions". It is proved that this modified algorithm is well-defined. It is also proved that every point of accumulation of the sequence generated by this algorithm is a second-order point of the nonlinear programming problem. Two different convergence proofs are given having in mind two interpretations of the presented algorithm.

Keywords: Forcing function, step-size algorithm, second-order conditions.

1. INTRODUCTION

We are concerned with the following problem of the unconstrained optimization:

\[ \min \{ \varphi(x) | x \in D \} \]  (1)

where \( \varphi : D \subset \mathbb{R}^n \rightarrow \mathbb{R} \) is a twicecontinuously differentiable function on an open set \( D \).

We consider iterative algorithms to find an optimal solution to problem (1) generating sequences of points \( \{x_k\} \) of the following form:

\[ x_{k+1} = x_k + \alpha_k s_k + \beta_k d_k, \quad k = 0, 1, \ldots, \]  (2)

\[ s_k, d_k \neq 0, \quad \langle \nabla \varphi(x_k), s_k \rangle \leq 0, \]  (3)

and the steps \( \alpha_k \) and \( \beta_k \) are defined by a particular step-size algorithm.

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Before we present the modified algorithm, we shall define the original second-order step-size algorithm.

The original McCormick-Armijo’s second order step-size algorithm \([4]\) defines \(\alpha_k\) in the following way: \(\alpha_k > 0\) is a number satisfying

\[
\alpha_k = 2^{-i(k)},
\]

where \(i(k)\) is the smallest integer from \(i = 0, 1, \ldots\), such that

\[
x_{k+1} = x_k + 2^{-i(k)} s_k + 2 \frac{\sigma}{2} d_k \in D
\]

and

\[
\varphi(x_k) - \varphi(x_{k+1}) \geq \gamma \left[ -\langle \nabla \varphi(x_k), s_k \rangle - \frac{1}{2} \{H(x_k)d_k, d_k\} \right] 2^{-i(k)},
\]

where \(0 < \gamma < 1\) is a preassigned constant, \(H(x)\) - the Hessian matrix of the function \(\varphi\) at \(x, s_k, d_k\) - direction vectors satisfying relations \((3)\).

We begin with the definition which we need in the following text.

**Definition** (See\([5]\)). A mapping \(\sigma : [0, \infty) \rightarrow [0, \infty)\) is a forcing function if for any sequence \(\{t_k\} \subset [0, \infty)\)

\[
\lim_{k \rightarrow \infty} \sigma(t_k) = 0 \quad \text{implies} \quad \lim_{k \rightarrow \infty} t_k = 0
\]

and \(\sigma(t) > 0\) for \(t > 0\).

(The concept of the forcing function was introduced first by Elkin in [3]).

### 2. A MODIFICATION OF THE SECOND-ORDER STEP-SIZE ALGORITHM

The modified algorithm defines \(\alpha_k\) in the following way: \(\alpha_k > 0\) is a number satisfying

\[
\alpha_k = q^{-i(k)}, \quad q > 1,
\]

where \(i(k)\) is the smallest integer from \(i = 0, 1, \ldots\), such that

\[
x_{k+1} = x_k + q^{-i(k)} s_k + q \frac{\sigma}{2} d_k \in D
\]

and

\[
\varphi(x_k) - \varphi(x_{k+1}) \geq q^{-i(k)} \left[ \sigma_1 (-\langle \nabla \varphi(x_k), s_k \rangle) + \sigma_2 \frac{1}{2} \{H(x_k)d_k, d_k\} \right]
\]
where \( \sigma_1 : [0, \infty) \to [0, \infty) \) and \( \sigma_2 : [0, \infty) \to [0, \infty) \) are the forcing functions such that 
\[
\delta_1 t \leq \sigma_1(t) \leq \delta_1^t, \quad \delta_2 t \leq \sigma_2(t) \leq \delta_2^t \quad 0 < \delta_1 < \delta_1^2 < 1, \quad 0 < \delta_2 < \delta_2^2 < 1
\]
and \( s_k, d_k \) are the direction vectors satisfying (3) and \( \langle H(x_k)d_k, d_k \rangle \leq 0 \).

In order to have a finite value \( i(k) \), it is sufficient that \( s_k \) and \( d_k \) satisfy (3) and, in addition, that
\[
\langle \nabla \phi(x_k), s_k \rangle < 0 \quad \text{whenever} \quad \nabla \phi(x_k) \neq 0 \tag{6A}
\]
and
\[
\langle H(x_k)d_k, d_k \rangle < 0 \quad \text{whenever} \quad \nabla \phi(x_k) = 0. \tag{6B}
\]

Now we shall prove the first convergence theorem.

**Theorem 1.** Let \( \phi : D \subset \mathbb{R}^n \to \mathbb{R} \) be a twicecontinuously differentiable function on the open set \( D \). Let the sequence \( \{x_k\} \) be defined by relations (2), (3), (4), (5), (6A) and (6B). Let \( \bar{x} \) be a point of accumulation of \( \{x_k\} \) and \( K_1 \) a set of indices such that \( x_k \to \bar{x} \) for \( k \in K_1 \).

Assume that:
1. the sequences \( \{s_k\} \) and \( \{d_k\} \) are uniformly bounded;
2. \(-\langle \nabla \phi(x_k), s_k \rangle \geq \mu_k(\| \nabla \phi(x_k) \|), \quad k \in K_1, \) where \( \mu_k : [0, \infty) \to [0, \infty), \ k \in K_1 \) are forcing functions;
3. there exists a value \( \beta > 0 \) such that
\[
-\langle H(x_k)d_k, d_k \rangle \geq \beta \langle H(x_k)e_k^{\min}, e_k^{\min} \rangle,
\]
where \( e_k^{\min} \) is an eigenvector of \( H(x_k) \) associated with its minimum eigenvalue.

Then \( \bar{x} \) is a stationary point, that is
\[
\nabla \phi(\bar{x}) = 0
\]
and \( H(\bar{x}) \) is a positive semidefinite matrix with at least one eigenvalue equal to zero.

**Proof:** There are two cases to consider.

a) The integers \( \{i(k)\} \) for \( k \in K_1 \) are uniformly bounded from above by some value \( I \).

Because of the descent property it follows that all points of the accumulation have the same function value and
\[
(0 \geq \phi(x_0) - \phi(\bar{x}) = \sum_{k \in K_1} [\phi(x_k) - \phi(x_{k+1})] \geq
\]
\[
\geq \sum_{k \in K_1} q^{-i(k)} \left[ \sigma_1(-\langle \nabla \phi(x_k), s_k \rangle) + \sigma_2 \left( -\frac{1}{2} \langle H(x_k)d_k, d_k \rangle \right) \right] \geq
\]
\[
\geq q^{-I} \sum_{k \in K_1} \left[ -\langle \nabla \phi(x_k), s_k \rangle - \frac{1}{2} \langle H(x_k)d_k, d_k \rangle \right] \quad (\delta = \max \{\delta_1, \delta_2\})
\]
\[ \geq q^{-i} \delta \sum_{k \in K_1} \left[ \mu_k \left( \| \nabla \varphi(x_k) \| \right) + \frac{1}{2} \beta \left( H(x_k) \theta_k^{\min} , e_k^{\min} \right) \right]. \]

Since \( \varphi(\bar{x}) \) is finite and since each term in the brackets is greater than, or equal to zero for each \( k \in K_1 \), it follows that \( \mu_k (\nabla \varphi(x_k)) \to 0 \Rightarrow \| \nabla \varphi(x_k) \| \to 0 \) (according to the definition of forcing functions) \( \Rightarrow \nabla \varphi(\bar{x}) = 0 \) and that \( \{ H(\bar{x}) \theta_k^{\min} , \bar{e}_k^{\min} \} = 0 \), where \( \bar{e}_k^{\min} \) is some accumulation point of \( \{ e_k^{\min} \} \) for \( k \in K_1 \).

b) There is a subset \( K_2 \subset K_1 \) such that \( \lim_{k \to \infty} i(k) = \infty \).

Because of the definition of \( i(k) \), then either
\[ x_k + q^{-i(k)+1} s_k + q^{-i(k)+1} d_k \in D \]
or
\[ \varphi(x_k) = \varphi \left( x_k + q^{-i(k)+1} s_k + q^{-i(k)+1} d_k \right) < \]
\[ < q^{-i(k)+1} \left[ \sigma_1 (-\langle \nabla \varphi(x_k), s_k \rangle ) + \sigma_2 \left( -\frac{1}{2} \left( H(x_k) d_k , d_k \right) \right) \right]. \]

If the former condition held infinitely often, then because
\[ x_k + q^{-i(k)+1} s_k + q^{-i(k)+1} d_k \to x, \quad k \in K_2, \]
it would follow that \( \bar{x} \) is on the boundary of \( D \). Since \( D \) is an open set, \( \bar{x} \notin D \), it contradicts the theorem hypothesis. Therefore, without the loss of generality (7) can be considered to hold for all \( k \in K_2 \).

Since \( \varphi \in C^2 \), and since the sequences \( \{ s_k \} \) and \( \{ d_k \} \) are assumed to be uniformly bounded, the left-hand side of inequality (7) can be written as
\[ -q^{-i(k)+1} \langle \nabla \varphi(x_k), s_k \rangle - q^{-i(k)+1} \langle \nabla \varphi(x_k), d_k \rangle - \]
\[ -\frac{1}{2} H(x_k) \left( q^{-i(k)+1} s_k + q^{-i(k)+1} d_k \right) q^{-i(k)+1} s_k + q^{-i(k)+1} d_k \]
\[ < q^{-i(k)+1} \left[ \sigma_1 (-\langle \nabla \varphi(x_k), s_k \rangle ) + \sigma_2 \left( -\frac{1}{2} \left( H(x_k) d_k , d_k \right) \right) \right] < \]
\[ < q^{-i(k)+1} \left[ \sigma_1 \langle \nabla \varphi(x_k), s_k \rangle - \sigma_2 \langle H(x_k) d_k , d_k \rangle \right]. \]

Combining terms and incorporating a term where appropriate into \( o(q^{-i(k)+1}) \) yields (using the fact that \( -\langle \nabla \varphi(x_k), s_k \rangle \geq 0 \) :}
\[ o(q^{-i(k)+1}) > q^{-i(k)+1} \left[ (-1 + \delta_1) \langle \nabla \varphi(x_k), s_k \rangle - (-\delta_2 + 1) \frac{1}{2} \langle H(x_k)d_k, d_k \rangle \right]. \]

Using the theorem hypothesis 3 we obtain
\[ o(q^{-i(k)+1}) > q^{-i(k)+1} \left[ (-1 + \delta_1) \langle \nabla \varphi(x_k), s_k \rangle + (-\delta_2 + 1) \frac{\beta}{2} \langle H(x_k)e_k^{\min}, e_k^{\min} \rangle \right] \]

Dividing by \( q^{-i(k)+1} \) yields
\[
\frac{o(q^{-i(k)+1})}{q^{-i(k)+1}} > (-1 + \delta_1) \langle \nabla \varphi(x_k), s_k \rangle + (-\delta_2 + 1) \frac{\beta}{2} \langle H(x_k)e_k^{\min}, e_k^{\min} \rangle \geq \nabla \varphi(x_k) + e_k^{\min}.
\]

Since each term is, according to the assumptions, greater than or equal to zero, taking the limit as \( k \to \infty \) for \( k \in K_2 \) yields
\[
\mu_k(\| \nabla \varphi(x_k) \|) \to 0 \Rightarrow \| \nabla \varphi(x_k) \| \to 0 \Rightarrow \nabla \varphi(x) = 0
\]
and
\[
\langle H(x_k)e_k^{\min}, e_k^{\min} \rangle \to \langle H(x)e_k^{\min}, e_k^{\min} \rangle = 0.
\]

To prove the second convergence theorem we shall follow Y. Amaya [1]. Namely, we are going to show that the trajectory
\[
f(t, x_k) = x_k + t^2 s_k + t d_k \quad (8)
\]
proposed by the presented algorithm (i.e. satisfying the relations (2), (3), (4), (5), (6A) and (6B)) and
\[
\langle \nabla \varphi(x_k), s_k \rangle < 0
\]
\[
\langle \nabla \varphi(x_k), d_k \rangle \leq 0 \quad (9)
\]
and
\[
\langle H(x_k)d_k, d_k \rangle = 0
\]
if \( H(x_k) \) is positive semidefinite, and
\[
\langle \nabla \varphi(x_k), s_k \rangle \leq 0
\]
\[
\langle \nabla \varphi(x_k), d_k \rangle \leq 0
\]
and
\[
\langle H(x_k)d_k, d_k \rangle < 0 \quad (10)
\]
if \( H(x_k) \) is not positive semidefinite, has the properties set out in Amaya's paper.

Firstly, we shall briefly present Amaya's algorithm [1].
Let $\phi: D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be a twice continuously differentiable function on the open set $D$ (i.e. $\phi \in C^2$) which we want to minimize, and $h: \mathbb{R}^+ \times D \rightarrow \mathbb{R}^n$ is a function such that, for all $x \in D$, $h(0,x) = x$. We suppose that for every $x \in D$, $h(t,x)$ is $C^2$ for $t \geq 0$.

Given $x \in D$, the function $h(t,k)$ describes a trajectory in $D \subset \mathbb{R}^n$ originating at $x$. The minimizing algorithm defines a sequence $\{x_k\}$ in the following way:

$$x_{k+1} = \begin{cases} x_k & \text{if } x_k \in M, \\ h(t_k, x_k) & \text{if } x_k \notin M, \end{cases}$$

(11)

where $M = \{x \in D | \nabla \phi(x) = 0 \text{ and } \langle H(x)p, p \rangle \geq 0, p \in \mathbb{R}^n \}$.

For $x \in D$, we define the $C^2$-class function $f_x: \mathbb{R}^+ \rightarrow \mathbb{R}^n$ by

$$f_x(t) = \phi[h(t,x)], \ t \in \mathbb{R}^+.$$  

This function is shown to satisfy

$$f_x'(0) = \langle \nabla \phi(x_k), h(0,x_k) \rangle \quad \text{and} \quad f_x''(0) = \langle H(x_k)h(0,x_k), h(0,x_k) + \langle \nabla \phi(x_k), \dot{h}(0,x_k) \rangle \rangle,$$

where $\dot{h}$ and $\ddot{h}$ denote respectively the first and second derivatives of $h$ with respect to $t$.

The following assumptions are made:

- **A1.** $L = \{x \in D | \phi(x) \leq \phi(x_0)\}$ is bounded;
- **A2.** $f_x'(0) \leq 0$ for all $x \notin M$;
- **A3.** if $x \notin M$ and $f_x'(0) = 0$, then $f_x''(0) < 0$.

Amaya in Theorem 3.1 in [1] proves the convergence of a subsequence of points of $\{x_k\}$ defined by (11) to $\bar{x} \in M$, provided that $\phi \in C^2$ and that assumptions A1, A2, A3 hold.

Now we can present the second convergence theorem for the modified McCormick-Armijo’s algorithm.

**Theorem 2.** Under assumptions A1, A2 and A3 every point of accumulation $\bar{x}$ of the sequence $\{x_k\}$ generated by the modified McCormick-Armijo’s algorithm and additionally, satisfying (9) and (10) belongs to $M$, that is, the second-order necessary conditions are satisfied at $\bar{x}$.

**Proof:** Let us suppose that $x_k \notin M$ for $k = 0, 1, 2, \ldots$. From the choice of $t_k = \alpha_k$ by relations (2), (3), (4), (5), (6A) and (6B) we have that $f_{x_k}(t_k) \leq f_{x_k}(0)$, i.e. the sequence $\{\phi(x_k)\}$ is decreasing; hence $\{x_k\} \subset L$. Due to the assumption A1, the sequence $\{x_k\}$ has a point of accumulation $\bar{x}$. 


For the trajectory (8) we have:
\[
\begin{align*}
f'_{x_k}(0) &= \langle \nabla \varphi(x_k), \dot{h}(0, x_k) \rangle, \quad \dot{h}(0, x_k) = d_k, \\
f''_{x_k}(0) &= \left\{ H(x_k) \dot{h}(0, x_k), \dot{h}(0, x_k) + \langle \nabla \varphi(x_k), \dot{h}(0, x_k) \rangle \right\}, \quad \dot{h}(0, x_k) = s_k, \quad \text{i.e.} \\
f_{x_k}^\perp(0) &= \langle \nabla \varphi(x_k), d_k \rangle, \\
f_{x_k}^\parallel(0) &= \left\{ H(x_k) d_k, d_k \right\} + \langle \nabla \varphi(x_k), s_k \rangle.
\end{align*}
\]

From (6A) it follows that the assumption A2 holds. Let us examine the assumption A3. Assuming \( f'_{x_k}(0) = 0 \), we have two cases:

a) if \( H(x_k) \) is positive semidefinite, by applying (9) to the relation (11), we obtain

\[ f''_{x_k}(0) < 0. \]

b) if \( H(x_k) \) is not positive semidefinite, by applying (10) to the relation (11), we obtain

\[ f_{x_k}^\perp(0) < 0. \]

Following Amaya's proof of theorem 3.1 in [1] we conclude that \( \exists \in M \).

### 3. CONCLUSION

Because of general assumptions on the objective function \( \varphi \), the modified algorithm can be used for solving a wide class of unconstrained optimization problems. Also, the choice of forcing functions \( \sigma_1(t) \) and \( \sigma_2(t) \), with the property

\[ \delta_1 t \leq \sigma_1(t) \leq \delta_1 t, \quad \delta_2 t \leq \sigma_2(t) \leq \delta_2 t, \quad 0 < \delta_1 < \delta_1 < 1, \quad 0 < \delta_2 < \delta_2 < 1 \]

is wide.

Finally, this modified algorithm can be used for solving constrained optimization problems (see [2]) when constraints are adequately considered.

### REFERENCES


