DUALITY FOR MULTIOBJECTIVE FRACTIONAL PROGRAMMING PROBLEMS INVOLVING $d$-TYPE-I -SET $n$ - FUNCTIONS

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Abstract: We establish duality results under generalized convexity assumptions for a multiobjective nonlinear fractional programming problem involving $d$-type-I $n$-set functions. Our results generalize the results obtained by Preda and Stancu-Minasian [24], [25].

Keywords: d-type-I set functions, multiobjective programming, duality results.

1. INTRODUCTION

Consider the multiobjective nonlinear fractional programming problem involving $n$-set functions
minimize \( F(S) = \left( \frac{F_i(S)}{G_i(S)} \right)_{i \in P} \) subject to

\[ H_j(S) \leq 0, \ j \in M, S = (S_1, \ldots, S_n) \in \Gamma^n \]

where \( \Gamma^n \) is the \( n \)-fold product of a \( \sigma \)-algebra \( \Gamma \) of subsets of a given set \( X \), \( M = \{1, 2, \ldots, m\} \), \( F_i, G_i, \ i \in P = \{1, 2, \ldots, p\} \), and \( H_j, j \in M \) are differentiable real-valued functions defined on \( \Gamma^n \) with

\[ F_i(S) \geq 0 \text{ and } G_i(S) > 0, \text{ for all } i \in P. \] (1)

Let \( S_0 = \{S | S \in \Gamma^n, H(S) \leq 0\} \) be the set of all feasible solutions to (P), where \( H = (H_1, \ldots, H_m) \).

The term “minimize” being used in Problem (P) is for finding efficient, weakly and properly efficient solutions.

A feasible solution \( S^0 \) to (P) is said to be an \textit{efficient solution} to (P) if there exists no other feasible solution \( S \) to (P) so that \( F_i(S) \leq F_i(S^0) \), for all \( i \in P \), with strict inequality for at least one \( i \in P \).

A feasible solution \( S^0 \) to (P) is said to be a \textit{weakly efficient solution} to (P) if there exists no other feasible solution \( S \) to (P) so that \( F_i(S) < F_i(S^0) \), for all \( i \in P \).

The analysis of optimization problems with set or \( n \)-set functions i.e. selection of measurable subsets from a given space, has been the subject of several papers. For a historical survey of optimality conditions and duality for programming problems involving set and \( n \)-set functions the reader is referred to Stancu-Minasian and Preda’s review paper [28]. These problems arise in various applications including fluid flow [3], electrical insulator design [8], regional design (districting, facility location, warehouse layout, urban planning etc.) [10], statistics [11], [21] and optimal plasma confinement [30]. The general theory for optimizing set functions was first developed by Morris [20]. Many results of Morris [20] are only confined to functions of a single set. Corley [9] started to give the concepts of partial derivatives and derivatives of real-valued \( n \)-set functions.

Starting from the methods used by Jeyakumar and Mond [12] and Ye [31], Suneja and Srivastava [29] defined some new classes of scalar or vector functions called \( d \)-\text{-type-I}, \( d \)-\text{-pseudo-type-I}, \( d \)-\text{-quasi-type-I} etc. for a multiobjective nondifferentiable programming problem and obtained necessary and sufficient optimality criteria. Also, they established duality between this problem and its Wolfe-type and Mond-Weir-type duals and obtained some duality results considering the concept of a weak minimum.

In particular, multiobjective fractional subset programming problems have been the focus of intense interest in the past few years, and resulted in many papers [1], [2], [4]-[7], [13]-[17], [22], [23], [28], [33]-[35].

In this paper we establish duality results under generalized convexity assumptions for a multiobjective nonlinear fractional programming problem involving
generalized \( d \)-type-1 \( n \)-set functions. Our results generalize the results obtained by Preda and Stancu-Minasian \([24],[25]\).

2. DEFINITIONS AND PRELIMINARIES

In this section we introduce the notation and definitions which will be used throughout the paper.

Let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space and \( \mathbb{R}_+^n \) its positive orthant, i.e.

\[
\mathbb{R}_+^n = \{ x = (x_j) \in \mathbb{R}^n, x_j \geq 0, \ j = 1,\ldots,n \}.
\]

For \( x = (x_1,\ldots,x_n), y = (y_1,\ldots,y_n) \in \mathbb{R}^n \) we put \( x \leq y \) iff \( x_i \leq y_i \) for each \( i \in M \); \( x \leq y \) iff \( x \leq y \) each \( i \in M \), with \( x \neq y \); \( x < y \) iff \( x_i < y_i \) for each \( i \in M \) while \( x \neq y \) is the negation of \( x < y \). We write \( x \in \mathbb{R}_+^n \) iff \( x \geq 0 \).

Let \((X,\Gamma,\mu)\) be a finite non-atomic measure space with \( L_1(X,\Gamma,\mu) \) separable, and let \( d \) be the pseudometric on \( \Gamma^n \) defined by:

\[
d(S,T) = \left[ \sum_{i=1}^{n} \mu^2(S_i \Delta T_i) \right]^{1/2}
\]

for \( S = (S_1,\ldots,S_n), T = (T_1,\ldots,T_n) \in \Gamma^n \), where \( \Delta \) denotes the symmetric difference. Thus \((\Gamma^n,d)\) is a pseudometric space, which will serve as the domain for most of the functions that will be used in this paper.

For \( h \in L_1(X,\Gamma,\mu) \), the integral \( \int h \, d\mu \) will be denoted by \( \langle h, I_S \rangle \), where \( I_S \) is the indicator (characteristic) function of \( S \in \Gamma \).

We next introduce the notion of differentiability for \( n \)-set functions. This was originally introduced by Morris \([20]\) for set functions and subsequently extended by Corley \([9]\) to \( n \)-set functions.

A function \( \varphi: \Gamma \to \mathbb{R} \) is said to be differentiable at \( S^0 \in \Gamma \) if there exist \( D\varphi(S^0) \in L_1(X,\Gamma,\mu) \), called the derivative of \( \varphi \) at \( S^0 \), and \( \psi: \Gamma \times \Gamma \to \mathbb{R} \) such that for each \( S \in \Gamma \),

\[
\varphi(S) = \varphi(S^0) + \langle D\varphi(S^0), I_S - I_{S^0} \rangle + \psi(S,S^0),
\]

where \( \psi(S,S^0) \) is \( o(d(S,S^0)) \), that is, \( \lim_{d(S,S^0) \to 0} \frac{\psi(S,S^0)}{d(S,S^0)} = 0 \).

A function \( F: \Gamma^n \to \mathbb{R} \) is said to have a partial derivative at \( S^0 = (S_1^0,\ldots,S_n^0) \) with respect to its \( k \)-th argument if the function

\[
\varphi(S_k) = F(S_1^0,\ldots,S_{k-1}^0,S_k^0,S_{k+1}^0,\ldots,S_n^0)
\]

has derivative \( D\varphi(S_k^0) \) and we define \( D_k F(S^0) = D\varphi(S_k^0) \). If \( D_k F(S^0), 1 \leq k \leq n \), all exist, then we put \( DF(S^0) = (D_1 F(S^0),\ldots,D_n F(S^0)) \).
A function $F: \Gamma^* \to \mathbb{R}$ is said to be differentiable at $S^0$ if there exist $DF(S^0)$ and $\psi: \Gamma^* \times \Gamma^* \to \mathbb{R}$ such that

$$F(S) = F(S^0) + \sum_{i=1}^{n} \left\{ D_i F(S^0), I_{S_i} - I_{S_i^0} \right\} + \psi(S, S^0),$$

where $\psi(S, S^0)$ is $o(d(S, S^0))$, for all $S \in \Gamma^*$.


**Definition 1.** [24] We say that $(F, G)$ is of $d$-type-I at $S^0 \in \Gamma^*$ if there exist functions $\alpha, \beta: \Gamma^* \times \Gamma^* \to \mathbb{R} \setminus \{0\}$, $i \in P, j \in M$, such that for all $S \in S_0$, we have

$$F_i(S) - F_i(S^0) \geq \alpha_i(S, S^0) \sum_{j=1}^{n} \left\{ D_j F_i(S^0), I_{S_j} - I_{S_j^0} \right\}, \quad i \in P$$

and

$$-H_j(S^0) \geq \beta_j(S, S^0) \sum_{i=1}^{n} \left\{ D_i H_j(S^0), I_{S_i} - I_{S_i^0} \right\}, \quad j \in M.$$  

We say that $(F, H)$ is of $d$-semistrictly type-I at $S^0$ if in the above definition we have $S \neq S^0$ and (2) is a strict inequality.

Now, we introduce

**Definition 2.** [32] A feasible solution $S^0$ to (P) is said to be a regular feasible solution if there exists $\hat{S} \in \Gamma^*$ such that

$$H_j(S^0) + \sum_{i=1}^{n} \left\{ D_i H_j(S^0), I_{S_i} - I_{S_i^0} \right\} < 0, \quad j \in M.$$  

Now, for each $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p_+$ we consider the parametric problem

minimize$(F_i(S) - \lambda_i G_i(S), \ldots, F_p(S) - \lambda_p G_p(S))$

subject to

$$H_j(S) \leq 0, \quad j \in M, S = (S_1, \ldots, S_n) \in \Gamma^*.$$  

The problem $(P_{\lambda})$ is equivalent to the problem (P) in the sense that for particular choices of $\lambda_i, \ i \in P$, the two problems have the same set of efficient solutions. This equivalence is stated in the following lemma which is well known in fractional programming [27].
Lemma 3. An \( S^0 \) is an efficient solution to \((P)\) if and only if is an efficient solution to \((P_{\lambda^0})\) with \( \lambda^0 = \frac{F_i(S^0)}{G_i(S^0)}, \ i = 1, ..., p. \)

In this paper the proofs of the duality results for Problem \( (P) \) will invoke the following necessary efficiency result for \((P_{\lambda^0})\) (see Zalmai [32], Theorem 3.2).

Theorem 4. [32] Let \( S^0 \) be a regular efficient (or weakly efficient) solution to \((P)\) and assume that \( F_i, G_i, \ i \in P \) and \( H_j, \ j \in M \), are differentiable at \( S^0 \). Then there exist \( u^0 \in \mathbb{R}^p_+, \sum_{i=1}^p u^0_i = 1, \ v^0 \in \mathbb{R}^m_+, \) and \( \lambda^0 \in \mathbb{R}^p_+ \) such that

\[
\sum_{i=1}^p \left( \sum_{j=1}^m u^0_i \left( D_i F_i(S^0) - \lambda^0_i D_i G_i(S^0) \right) + \sum_{j=1}^m v^0_j D_j H_j(S^0), I_{\lambda_i} - I_{\lambda_i} \right) \geq 0, \text{ for all } S \in \Gamma^0 \tag{4}
\]

\[
u^0_i (F_i(S^0) - \lambda^0_i G_i(S^0)) \geq 0, \ i \in P \tag{5}
\]

\[
u^0_j H_j(S^0) = 0, \ j \in M. \tag{6}
\]

3. DUALITY

In this section, in the differentiable case, based on the equivalence of \((P)\) and \((P_{\lambda^0})\) a dual for \((P_{\lambda^0})\) is defined and some duality results in \( d\)-type-I assumptions are stated. With \((P_{\lambda^0})\) we associate a dual stated as

\[
\text{maximize} \ (\lambda_1, \ldots, \lambda_p) \tag{D}
\]

subject to

\[
\sum_{i=1}^p \left( \sum_{j=1}^m u_j \left( D_i F_i(T) - \lambda_i D_i G_i(T), I_{\lambda_i} - I_{\lambda_i} \right) + \sum_{j=1}^m v_j D_j H_j(T), I_{\lambda_i} - I_{\lambda_i} \right) \geq 0, \quad S \in \Gamma^0 \tag{7}
\]

\[
u_i (F_i(T) - \lambda_i G_i(T)) \geq 0, \quad i \in P, \tag{8}
\]

\[
u_j H_j(T) \geq 0, \quad j \in M, \tag{9}
\]

\[
u \in \mathbb{R}^m_+, \sum_{i=1}^p u_i = 1, \nu \in \mathbb{R}^m_+, \lambda \in \mathbb{R}^p_+. \tag{10}
\]

Let \( D_0 \) be the set of feasible solutions to \( (D) \). Let us prove the duality theorems.

Theorem 5. (Weak duality) Let \( S \) and \( (T, u, v, \lambda) \) be feasible solutions to problem \((P)\) and \((D)\), respectively such that \( (i,j) \) for each \( i \in P \) and \( j \in M \), \( (F_i(\cdot) - \lambda_i G_i(\cdot), H_j(\cdot)) \) is
of $d$-type-I at $T$; (i2) $u_i > 0$ for any $i \in P$, and for some $i \in P$ and $j \in M$, $(F_i(\cdot) - \lambda G_i(\cdot), H_j(\cdot))$ is of $d$-semistrictly type-I at $T$.

Then for any $S \in S_0$ one cannot have

$$\frac{F_i(S)}{G_i(S)} \leq \lambda_i \text{ for any } i \in P,$$

$$\frac{F_j(S)}{G_j(S)} < \lambda_j \text{ for some } j \in P. \tag{11} \tag{12}$$

**Proof:** Let us suppose the contrary that (11) and (12) hold. Then there exists $S$, a feasible solution for $(P_\lambda)$, such that (11) and (12) hold.

If hypothesis (i2) holds, then $u_i > 0$ for any $i = 1, \ldots, p$. From (1), (11) and (12) we get

$$\sum_{i=1}^{n} u_i (F_i(S) - \lambda_i G_i(S)) < 0. \tag{13}$$

Using the feasibility of $S$, and the relations (9) and (10), we have

$$v_j H_j(S) \leq 0 \leq v_j H_j(T) \forall j = 1, \ldots, m. \tag{14}$$

Since $\alpha_i(S,T) > 0, i \in P$, and $\beta_j(S,T) > 0, j \in M$, combining (8), (13) and (14) we obtain

$$\sum_{i=1}^{n} \frac{u_i}{\alpha_i(S,T)} (F_i(S) - \lambda_i G_i(S)) < \sum_{i=1}^{n} \frac{u_i}{\alpha_i(S,T)} (F_i(T) - \lambda_i G_i(T)) \tag{15}$$

$$+ \sum_{i=1}^{n} \frac{v_j H_j(T)}{\beta_j(S,T)}.$$

We claim that $S \neq T$ for if it is not true, then, from $u_i > 0, i \in P$, the feasibility of $S$ and (8) we obtain a contradiction with (11) and (12).

One the other hand, from $S \neq T$, (i1) and (i2), it follows that

$$(F_i(S) - \lambda_i G_i(S)) - (F_i(T) - \lambda_i G_i(T)) \geq$$

$$\alpha_i(S,T) \sum_{k=1}^{n} \{ D_k F_i(T) - \lambda D_k G_i(T), I_{k_i} - I_{i_k} \} \tag{16}$$

for any $i \in P$, with strict inequality for some $i$, and

$$-H_j(T) \geq \beta_j(S,T) \sum_{k=1}^{n} \{ D_k H_j(T), I_{k_j} - I_{j_k} \}, j \in M. \tag{17}$$

By dividing by $\alpha_i(S,T) > 0$ and $\beta_j(S,T) > 0$, respectively, the above inequalities reduce to the following
\[
\frac{F_i(S) - \lambda_i G_i(S)}{\alpha_i(S,T)} - \frac{F_i(T) - \lambda_i G_i(T)}{\alpha_i(S,T)} \geq \sum_{i=1}^{n} \left\{ D_i F_i(T) - \lambda_i D_i G_i(T), I_{S_i} - I_{T_i} \right\}
\]

(18)

for any \( i \in P \), with strict inequality for some \( i \), and

\[
-\frac{H_j(T)}{\beta_j(S,T)} \geq \sum_{i=1}^{n} \left\{ D_j H_j(T), I_{S_j} - I_{T_j} \right\}, \quad j \in M
\]

(19)

Multiplying the inequality (18) by \( u_i > 0 \), \( \forall i \in P \), and (19) by \( v_j \geq 0 \), \( \forall j \in M \), and summing after all \( i \) and \( j \), respectively, yields

\[
\sum_{i=1}^{n} u_i \left\{ D_i F_i(T) - \lambda_i D_i G_i(T), I_{S_i} - I_{T_i} \right\} + \sum_{j=1}^{m} v_j \left\{ D_j H_j(T), I_{S_j} - I_{T_j} \right\} > 0
\]

(20)

Now, by (15) it follows

\[
\sum_{i=1}^{n} u_i \left\{ D_i F_i(T) - \lambda_i D_i G_i(T), I_{S_i} - I_{T_i} \right\} + \sum_{j=1}^{m} v_j \left\{ D_j H_j(T), I_{S_j} - I_{T_j} \right\} < 0
\]

This inequality contradicts (7). Thus the theorem is proved.

**Corollary 6.** Let \( S^0 \) and \((S^0, u^0, v^0, \lambda^0)\) be feasible solutions to \((P_x)\) and \((D)\), respectively. If the hypotheses of Theorem 5 are satisfied, then \( S^0 \) is an efficient solution to \((P_x)\) and \((S^0, u^0, v^0, \lambda^0)\) is an efficient solution to \((D)\).

**Proof:** We proceed by contradiction. If \( S^0 \) is not an efficient solution to \((P_x)\) then there exists a feasible solution \( S' \) to \((P_x)\) such that

\[
F_i(S') \leq \lambda_i G_i(S'), \quad \forall i \in P,
\]

and

\[
F_i(S') < \lambda_i^0 G_i(S'), \quad \text{for some } j \in P.
\]

(21)

Since \((S^0, u^0, v^0, \lambda^0)\) is a feasible solution to \((D)\) by (21), and Theorem 5 we obtain a contradiction. Hence \( S^0 \) is an efficient solution to \((P_x)\). In the same way we obtain that \((S^0, u^0, v^0, \lambda^0)\) is an efficient solution to \((D)\).

**Theorem 7.** *(Strong duality)* Let \( S^0 \) be a regular efficient solution to \((P)\). Then there exist \( u^0 \in \mathbb{R}^n_+ \), \( \sum_{i=1}^{n} u^0_i = 1 \), \( v^0 \in \mathbb{R}^m_+ \), and \( \lambda^0 \in \mathbb{R}^p_+ \), such that \((S^0, u^0, v^0, \lambda^0)\) is a feasible solution to \((D)\). Further, if the conditions of Weak Duality Theorem 5 also hold, then
\((S^0, u^0, v^0, \lambda^0)\) is an efficient solution to (D) and the values of the objective functions of (P) and (D) are equal at \(S^0\) and \((S^0, u^0, v^0, \lambda^0)\) respectively.

**Proof:** Using Theorem 4 we obtain that there exist \(v^0 \in \mathbb{R}_{+}^n, \sum_{i=1}^{n} v_i^0 = 1, v^0 \in \mathbb{R}_{+}^n\), and (4) and (5) hold. Thus, \((S^0, u^0, v^0, \lambda^0)\) satisfies (7) – (10). Hence, \((S^0, u^0, v^0, \lambda^0)\) is a feasible solution to (D). Further, if Theorem 5 holds then, by Corollary 6 we obtain that this solution \((S^0, u^0, v^0, \lambda^0)\) is also an efficient solution to (D), and the values of the objective functions of (P) and (D) are equal at \(S^0\) and \((S^0, u^0, v^0, \lambda^0)\) respectively.

Now we give a strict converse duality theorem of Mangasarian type [19] for (P\(_1\)) and (D).

**Theorem 8. (Strict converse duality)** Let \(S^0\) and \((S^0, u^0, v^0, \lambda^0)\) be efficient solutions to (P\(_1\)) and (D), respectively. Assume that

\[
(j_1) \sum_{i=1}^{n} \frac{u_i^0}{\alpha_i(S^0, S^0)}(F_i(S^0) - \lambda^0_i G_i(S^0)) \leq \sum_{i=1}^{n} \frac{u_i^0}{\alpha_i(S^0, S^0)}(F_i(S^0) - \lambda^0_i G_i(S^0));
\]

\[
(j_2) \text{for any } i \in P \text{ and } j \in M, \quad (F_i(\cdot) - \lambda^0_i G_i(\cdot), H_j(\cdot)) \text{ is of } d\text{-semistrictly type-I at } S^0. \quad \text{Then, } S^0 = S^*.
\]

**Proof:** We assume that \(S^0 \neq S^*\) and exhibit a contradiction. Using \((j_2)\) we obtain

\[
(F_i(S^0) - \lambda^0_i G_i(S^0)) - (F_i(S^0) - \lambda^0_i G_i(S^0)) > \alpha_i(S^0, S^0) \sum_{k=1}^{n} \left\{ D_k F_i(S^0) - \lambda^0_i D_k G_i(S^0), I_{S^0} - I_{S^0} \right\}
\]

for any \(i \in P\) and

\[
-H_i(S^0) \geq \beta_i(S^*, S^0) \sum_{k=1}^{n} \left\{ D_i H_j(T), I_{S^0} - I_{S^0} \right\}, \quad j \in M.
\]

By dividing by \(\alpha_i(S^0, S^0) > 0\) and \(\beta_i(S^*, S^0) > 0\), respectively, the above inequalities reduce to the following

\[
\frac{F_i(S^0) - \lambda^0_i G_i(S^0)}{\alpha_i(S^0, S^0)} - \frac{F_i(S^0) - \lambda^0_i G_i(S^0)}{\alpha_i(S^0, S^0)} > \sum_{k=1}^{n} \left\{ D_k F_i(S^0) - \lambda^0_i D_k G_i(S^0), I_{S^0} - I_{S^0} \right\}
\]

(22)

for any \(i \in P\) and

\[
-\frac{H_i(S^0)}{\beta_i(S^*, S^0)} \geq \sum_{k=1}^{n} \left\{ D_i H_j(S^0), I_{S^0} - I_{S^0} \right\}, \quad j \in M.
\]

(23)
Multiplying the inequality (22) by $u^0_i \geq 0$, $\forall i \in P$, and (23) by $v^0 \geq 0$, $\forall j \in M$, and summing after all $i$ and $j$, respectively, yields

$$
\sum_{i=1}^{p} \frac{u^0_i}{\alpha_i(S^*, S^0)} (F_i(S^*) - \lambda^0_i G_i(S^*)) - \sum_{i=1}^{p} \frac{u^0_i}{\alpha_i(S^*, S^0)} (F_i(S^0) - \lambda^0_i G_i(S^0))
$$

$$
- \sum_{j=1}^{m} \frac{v^0_j H(S_j)}{\beta_j(S^*, S^0)} > \sum_{i=1}^{p} \sum_{k=1}^{n} v^0_j \left( D_j F_i(S^0) - \lambda^0_j D_j G_i(S^0), I_{S_j} - I_{S^0} \right)
$$

(24)

$$
\sum_{j=1}^{m} \sum_{k=1}^{n} v^0_j \left( D_j H_j(S^0), I_{S_j} - I_{S^0} \right).
$$

Now, because $(S^0, u^0, v^0, \lambda^0)$ is a feasible solution to (D) by (7) we get

$$
\sum_{i=1}^{p} \frac{u^0_i}{\alpha_i(S^*, S^0)} (F_i(S^*) - \lambda^0_i G_i(S^*)) - \sum_{i=1}^{p} \frac{u^0_i}{\alpha_i(S^*, S^0)} (F_i(S^0) - \lambda^0_i G_i(S^0)) -
$$

$$
- \sum_{j=1}^{m} \frac{v^0_j H(S^0)}{\beta_j(S^*, S^0)} > 0.
$$

(25)

Since $v^0_j H_j(S^0) \geq 0$ for any $j \in M$, by (25) we obtain

$$
\sum_{i=1}^{p} \frac{u^0_i}{\alpha_i(S^*, S^0)} (F_i(S^*) - \lambda^0_i G_i(S^*)) > \sum_{i=1}^{p} \frac{u^0_i}{\alpha_i(S^*, S^0)} (F_i(S^0) - \lambda^0_i G_i(S^0))
$$

which contradicts the assumption $(j_1)$. Thus the theorem is proved.

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