INVENTORY MODELS WITH STOCK- AND PRICE-DEPENDENT DEMAND FOR DETERIORATING ITEMS BASED ON LIMITED SHELF SPACE

Chun-Tao CHANG, Yi-Ju CHEN, Tzong-Ru TSAI and Shuo-Jye WU

Department of Statistics
Tamkang University
Tamsui, Taipei

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Abstract: This paper deals with the problem of determining the optimal selling price and order quantity simultaneously under EOQ model for deteriorating items. It is assumed that the demand rate depends not only on the on-display stock level but also the selling price per unit, as well as the amount of shelf/display space is limited. We formulate two types of mathematical models to manifest the extended EOQ models for maximizing profits and derive the algorithms to find the optimal solution. Numerical examples are presented to illustrate the models developed and sensitivity analysis is reported.

Keywords: Inventory control, pricing, stock-dependent demand, deterioration.

1. INTRODUCTION

In the classical inventory models, the demand rate is regularly assumed to be either constant or time-dependent but independent of the stock levels. However, practically an increase in shelf space for an item induces more consumers to buy it. This occurs owing to its visibility, popularity or variety. Conversely, low stocks of certain goods might raise the perception that they are not fresh. Therefore, it is observed that the demand rate may be influenced by the stock levels for some certain types of inventory. In years, marketing researchers and practitioners have recognized the phenomenon that the demand for some items could be based on the inventory level on display. Levin et al. (1972) pointed out that large piles of consumer goods displayed in a supermarket would attract the customer to buy more. Silver and Peterson (1985) noted that sales at the retail

As shown in Levin et al. (1972), “large piles of consumer goods displayed in a supermarket will lead customers to buy more. Yet, too many goods piled up in everyone’s way leave a negative impression on buyers and employees alike.” Hence, in this present paper, we first consider a maximum inventory level in the model to reflect the facts that most retail outlets have limited shelf space and to avoid a negative impression on customer because of excessively piled up in everyone’s way. Since the demand rate not only is influenced by stock level, but also is associated with selling price, we also take into account the selling price and then establish an EOQ model in which the demand rate is a function of the on-display stock level and the selling price. In Section 2, we provide the fundamental assumptions for the proposed EOQ model and the notations used throughout this paper. In Section 3, we set up a mathematical model. The properties of the optimal solution are discussed as well as its solution algorithm and numerical examples are presented. In Section 4, an optimal ordering policy with selling price predetermined is investigated. Theorems 1 and 2 are provided to show the characteristics of the optimal solution. An easy-to-use algorithm is developed to determine the optimal cycle time, economic order quantity and ordering point. Finally, we draw the conclusions and address possibly future work in Section 5.

2. ASSUMPTIONS AND NOTATIONS

A single-item deterministic inventory model for deteriorating items with price- and stock-dependent demand rate is presented under the following assumptions and notations.
1. Shortages are not allowed to avoid lost sales.
2. The maximum allowable number of displayed stocks is $B$ to avoid a negative impression and due to limited shelf/display space.
3. Replenishment rate is infinite and lead time is zero.
4. The fixed purchasing cost $K$ per order is known and constant.
5. Both the purchase cost $c$ per unit and the holding cost $h$ per unit per unit time are known and constant. The constant selling price $p$ per unit is a decision variable within the replenishment cycle, where $p > c$.
6. The constant deterioration rate $\theta$ ($0 \leq \theta < 1$) is only applied to on-hand inventory. There are two possible cases for the cost of a deteriorated item $s$: (1) if there is a salvage value, that value is negative or zero; and (2) if there is a disposal cost, that value is positive. Note that $c > s$ (or $-s$).
7. All replenishment cycles are identical. Consequently, only a typical planning cycle with $T$ length is considered (i.e., the planning horizon is $[0, T]$).
8. The demand rate $R(I(t), p)$ is deterministic and given by the following expression:

$$R(I(t), p) = \alpha(p) + \beta I(t),$$

where $I(t)$ is the inventory level at time $t$, $\beta$ is a non-negative constant, and $\alpha(p)$ is a non-negative function of $p$ with $\alpha'(p) = d\alpha(p)/dp < 0$.

9. As stated in Urban (1992), “it may be desirable to order large quantities, resulting in stock remaining at the end of the cycle, due to the potential profits resulting from the increased demand.” Consequently, the initial and ending inventory levels $y$ are not restricted to be zero (i.e., $y \geq 0$). The order quantity $Q$ enters into inventory at time $t = 0$. Consequently, $I(0) = Q + y$. During the time interval $[0, T]$, the inventory is depleted by the combination of demand and deterioration. At time $T$, the inventory level falls to $y$, i.e., $I(T) = y$. The initial and ending inventory level $y$ can be called ordering point.

The mathematical problem here is to determine the optimal values of $T$, $p$ and $y$ such that the average net profit in a replenishment cycle is maximized.

3. MATHEMATICAL MODEL AND ANALYSIS

At time $t = 0$, the inventory level $I(t)$ reaches the top $\bar{I}$ (with $\bar{I} \leq B$) due to ordering the economic order quantity $Q$. The inventory level then gradually depletes to $y$ at the end of the cycle time $t = T$ mainly for demand and partly for deterioration. A graphical representation of this inventory system is depicted in Figure 1. The differential equation expressing the inventory level at time $t$ can be written as follows:
Figure 1. Graphical Representation of Inventory System

\[ I'(t) + \theta I(t) = -R(I(t), p), \quad 0 \leq t \leq T, \quad (1) \]

with the boundary condition \( I(T) = y \). Accordingly, the solution of Equation (1) is given by

\[ I(t) = ye^{(\theta+\beta)(T-t)} + \frac{\alpha(p)}{\theta+\beta} (e^{(\theta+\beta)(T-t)} - 1), \quad 0 \leq t \leq T. \quad (2) \]

Applying (2), we obtain that the total profit \( TP \) over the period \([0, T]\) is denoted by

\[ TP = (p-c) \int_0^T R(I(t), p) \, dt - K - [h + \theta (c + s)] \int_0^T I(t) \, dt \]

\[ = (p-c)\alpha(p) T - K + [(p-c)\beta - h - \theta (c + s)] \times \]

\[ \left[ \int_0^T ye^{(\theta+\beta)(T-t)} + \frac{\alpha(p)}{\theta+\beta} (e^{(\theta+\beta)(T-t)} - 1) \, dt \right] \]

\[ = (p-c)\alpha(p) T - K + [(p-c)\beta - h - \theta (c + s)] \times \]

\[ \left[ \frac{1}{\theta+\beta} \left( y + \frac{\alpha(p)}{\theta+\beta} \left( e^{(\theta+\beta)T} - 1 \right) - \frac{\alpha(p)}{\theta+\beta} T \right) \right]. \quad (3) \]

Hence, the average profit per unit time is

\[ AP = \frac{TP}{T} \]
\( = (p - c)\alpha(p) + \{ - K + [(p - c)\beta - h - \theta(c + s)] \times \)
\[
\left[ \frac{1}{\theta + \beta} \left( \frac{\alpha(p)}{\theta + \beta} \right) e^{(\theta + \beta)T} - 1 - \frac{\alpha(p)}{\theta + \beta} T \right] \}
\]. (4)

Necessary conditions for an optimal solution

Taking the first derivative of \( AP \) as defined in (4) with respect to \( T \), we have
\[
\frac{\partial AP}{\partial T} = \frac{1}{T^2} \{ K + [(p - c)\beta - h - \theta(c + s)] \times \}
\[
\left( \frac{1}{\theta + \beta} \right) \left( \frac{\alpha(p)}{\theta + \beta} \right) \{ (\theta + \beta)Te^{(\theta + \beta)T} - e^{(\theta + \beta)T} + 1 \}. \] (5)

From Appendix 1, we show that \( \{ (\theta + \beta)Te^{(\theta + \beta)T} - e^{(\theta + \beta)T} + 1 \} \) is greater than zero. \( \{ (p - c)\beta \} \) is the benefit received from a unit of inventory and \( \{ h + \theta(c + s) \} \) is the total cost (i.e., holding and deterioration costs) per unit inventory. Let \( \Delta_1 = (p - c)\beta \) and \( \Delta_2 = h + \theta(c + s) \), based on the values of \( \Delta_1 \) and \( \Delta_2 \), two distinct cases for finding the optimal \( T^* \) are discussed as follows:

Case 3.1 \( \Delta_1 \geq \Delta_2 \) (Building up inventory is profitable)

“\( \Delta_1 \geq \Delta_2 \)” implies that the benefit received from a unit of inventory is larger than the total cost (i.e., holding and deterioration costs) due to a unit of inventory. That is, it is profitable to build up inventory. Using Appendix 1, \( \frac{\partial AP}{\partial T} > 0 \), if \( \Delta_1 \geq \Delta_2 \). Namely, \( AP \) is an increasing function of \( T \) with \( I(t) \leq B \). Therefore, we should pile up inventory to the maximum allowable number \( B \) of stocks displayed in a supermarket without leaving a negative impression on customers. So, \( I(0) = B \). From \( I(0) = B \), we know
\[
T = \frac{1}{\theta + \beta} \ln \left( \frac{B(\theta + \beta) + \alpha(p)}{y(\theta + \beta) + \alpha(p)} \right). \] (6)

which implies that \( T \) is a function of \( p \) and \( y \).

Substituting (6) into (4), we know that \( AP \) is a function of \( y \) and \( p \).

The necessary conditions of \( AP \) to be maximized are \( \frac{\partial AP}{\partial y} = 0 \) and \( \frac{\partial AP}{\partial p} = 0 \). Hence, we have the following two conditions:
\[
\frac{-K(\theta + \beta)^2}{\Delta_1 - \Delta_2} = (\theta + \beta)(y - B) + [\alpha(p) + y(\theta + \beta)] \ln \left( \frac{\alpha(p) + B(\theta + \beta)}{\alpha(p) + y(\theta + \beta)} \right), \] (7)

and
From (7) and (8), the optimal values of \( p^* \) and \( y^* \) are obtained. Substituting \( p^* \) and \( y^* \) into (6), the optimal value \( T^* \) is solved. Since \( AP(y, p) \) is a complicated function, it is not possible to show analytically the validity of the sufficient conditions. However, according to the following mention, we know that the optimal solution can be obtained by numerical examples. Because building up is profitable and \( AP \) is a continuous function of \( y \) and \( p \) over the compact set \([0, B] \times [0, L] \), where \( L \) is a sufficient large number, so \( AP \) has a maximum value. It is clear that \( AP \) is not maximum at \( y = 0 \) (or \( B \)) and \( p = 0 \) (or \( L \)). Therefore, the optimal solution is an inner point and must satisfy (7) and (8). If the solution from (7) and (8) is unique, then it is the optimal solution. Otherwise, we have to substitute them into (4) and find the one with the largest values.

**Case 3.2.** \( \Delta_1 < \Delta_2 \) (Building up inventory is not profitable)

First taking the partial derivative of \( AP \) with respect to \( y \), we obtain

\[
\frac{\partial AP}{\partial y} = \frac{1}{T} \left[ (\Delta_1 - \Delta_2) \frac{1}{\theta + \beta} (e^{(\theta + \beta)T} - 1) \right] < 0.
\]  

(10)

Next, we get \( y^* = 0 \). Substituting \( y^* = 0 \) into (4), we have \( AP \) is a function of \( p \) and \( T \).

So, the necessary conditions of \( AP \) to be maximized are \( \frac{\partial AP}{\partial T} = 0 \) and \( \frac{\partial AP}{\partial p} = 0 \). Then, we get the following two conditions:

\[
-\frac{K(\theta + \beta)^2}{\alpha(p)(\Delta_1 - \Delta_2)} = (\theta + \beta)Te^{(\theta + \beta)T} - e^{(\theta + \beta)T} + 1,
\]  

(11)

and

\[
[ \alpha(p)\theta + (p \theta + h + \theta s)\alpha'(p) ]T = -\frac{(e^{(\theta + \beta)T} - 1)}{\theta + \beta} \left[ \beta \alpha(p) + (\Delta_1 - \Delta_2) \alpha'(p) \right].
\]  

(12)
From (11) and (12), we can obtain the values for $T$ and $p$. Substituting $y^* = 0$, $T$ and $p$ into (2) and check whether $I(0) < B$ or not. If $I(0) < B$, then the optimal values $T^* = T$, $p^* = p$ and $Q^* = I(0)$. If $I(0) \geq B$, then set $I(0) = B$ and obtain

$$T = \frac{1}{\theta + \beta} \ln \left( \frac{B(\theta + \beta) + \alpha(p)}{\alpha(p)} \right), \quad (13)$$

which is a function of $p$. Substituting $y^* = 0$ and (13) into (4), we have $AP$ is only depend on $p$. Then, the necessary conditions of $AP$ to be maximized is $\frac{dAP}{dp} = 0$. Hence,

$$\frac{\alpha(p)\theta + \theta(p + s) + h\alpha'(p)}{\theta + \beta} T^2 + \frac{(\epsilon^{(\theta+\beta)T} - 1)}{\theta + \beta} \left[ \frac{\alpha(p)}{\theta + \beta} + (\Delta_1 - \Delta_2) \frac{\alpha'(p)}{\theta + \beta} \right] T$$

$$= - \{K + (\Delta_1 - \Delta_2) \frac{\alpha(p)}{(\theta + \beta)} \left[ ((\theta + \beta)T \epsilon^{(\theta+\beta)T} - \epsilon^{(\theta+\beta)T} + 1) \right] \frac{dT}{dp}, \quad (14)$$

where $T$ is defined as (13) and

$$\frac{dT}{dp} = \frac{-B\alpha'(p)}{[\alpha(p) + (\theta + \beta)B\alpha(p)]. \quad (15)}$$

The optimal value $p^*$ is determined by (14). Substituting $p^*$ into (13), the optimal value $T^*$ is solved.

**Algorithm**:

The algorithm for determining an optimal selling price $p^*$, optimal ordering point $y^*$, optimal cycle time $T^*$, and optimal economic order quantity $Q^*$ is summarized as follows:

**Step 1.** Solving (7) and (8), we get the values for $p$ and $y$.

**Step 2.** If $\Delta_1 \geq \Delta_2$, then $p^* = p$, $y^* = y$, $Q^* = B - y^*$, and the optimal value $T^*$ can be obtained by substituting $p$ and $y$ into (6).

**Step 3.** If $\Delta_1 < \Delta_2$, then re-set $y^* = 0$. By solving (11) and (12), we get the values for $T$ and $p$. Substituting $y^* = 0$, $p$ and $T$ into (2) to find $I(0)$. If $I(0) < B$, then the optimal values $T^* = T$, $p^* = p$ and $Q^* = I(0)$, and stop. Otherwise, go to Step 4.

**Step 4.** If the simultaneous solutions $T$ and $p$ in (11) and (12) make $I(0) > B$, then the optimal value $p^*$ is determined by (14), $T^*$ is obtained by substituting $p^*$ into (13), and $Q^* = I(0)$ by substituting $p^*$ and $T^*$ into (2).

**Numerical examples**

To illustrate the proposed model, we provide the numerical examples here. For simplicity, we set the function $\alpha(p) = xp^{-r}$, where $x$ and $r$ are non-negative constants. That is, we assume that demand is a constant elasticity function of the price.
Example 3.1 Let $K = 10$ per cycle, $x = 1000$ units per unit time, $h = 0.5$ per unit per unit time, $s = 0$ per unit, $r = 2.5$ and $\theta = 0.05$. Following through the proposed algorithm, the optimal solution can be obtained. Since (4) and (6)-(9) are nonlinear, they are extremely difficult to solve. We use Maple 9.5 software to solve them. The computational results for the optimal values of $p, y, T, Q$ and $AP$ with respect to different values of $\beta, B, c$ are shown in Table 3.1.

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<th>$y^*$</th>
<th>$Q^*$</th>
<th>$p^*$</th>
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<th>$AP^*$</th>
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Based on the computational results as shown in Table 3.1, we obtain the following managerial phenomena when building up inventory is profitable:

1. A higher value of $\beta$ causes higher values of $Q^*$ and $AP^*$, but lower values of $y^*$, $p^*$ and $T^*$. It reveals that the increase of demand rate will result in the increases of optimal economic order quantity and average profit, but the decreases of optimal ordering point, selling price and cycle time.

2. A higher value of $B$ causes higher values of $Q^*$, $T^*$ and $AP^*$, but lower values of $y^*$ and $p^*$. It implies that the increase of shelf space will result in the increases of optimal economic order quantity, cycle time and average profit, but the decreases of optimal ordering point and selling price.

3. A higher value of $c$ causes higher values of $Q^*$ and $T^*$, but lower values of $y^*$ and $AP^*$. It implies that the increase of purchase cost will result in the increases of optimal economic order quantity and cycle time, but the decreases of optimal ordering point and average profit.

Example 3.2 Let $K = 10$ per cycle, $x = 1000$ units per unit time, $h = 0.2$ per unit per unit time, $c = 1.0$ per unit, $s = 0$ per unit, $r = 2.8, \theta = 0.05$ and $B = 300$. From Step 3 of the proposed algorithm, we obtain the optimal ordering point $y^* = 0$. Using Maple 9.5 software, we solve (2), (4), (11) and (12). The computational results for the optimal values of $p, Q, T$ and $AP$ with respect to different values of $\beta$ are shown in Table 3.2.
Table 3.2 shows that a higher value of $\beta$ causes higher values of $Q^*$, $p^*$, $T^*$ and $AP^*$. It indicates that the increase of demand rate will result in the increases of optimal economic order quantity, selling price, cycle time and average profit, when building up inventory is not profitable.

### 4. AN OPTIMAL ORDERING POLICY MODEL WITH SELLING PRICE PREDETERMINED

In the previous section, only the necessary condition was outlined for determining optimal values of $p$, $T$, $Q$ and $y$. The existence and uniqueness of the optimal solution remained unexplored. In addition, most firms have no pricing power in today’s business competition. As a result, most firms are not able to change price. In order to reflect this important fact, in this section, we study a special case that the selling price is predetermined. In this special case, we are able to show that the optimal solution to the relevant problem exists uniquely. Theorems 1 and 2 are provided to present the characteristics of the optimal solution. An easy-to-use algorithm is proposed to determine the optimal cycle time, ordering point and order quantity.

**Necessary conditions for an optimal solution**

Since $p$ is predetermined, $\alpha(p)$ is reduced to $\alpha$. Equation (4) can be rewritten as follows:

$AP = (p - c)\alpha + \{-K + (\Delta_1 - \Delta_2)\times$

$\left[\frac{1}{\theta + \beta} \left(y + \frac{\alpha}{\theta + \beta} \right) \left(e^{(\theta + \beta)T} - 1\right) - \frac{\alpha}{\theta + \beta} \right]/T.$ \hspace{1cm} (16)

Evidently, $AP$ is a function of $T$ and $y$. The model now is to determine the optimal values of $T$ and $y$ such that $AP$ in (16) is maximized.

Taking the first derivative of $AP$ with respect to $T$, we have

$\frac{\partial AP}{\partial T}$

$= \frac{1}{T^2} \left(K + (\Delta_1 - \Delta_2) \left(\frac{1}{\theta + \beta} \right) \left(y + \frac{\alpha}{\theta + \beta} \right) \left( (\theta + \beta)Te^{(\theta + \beta)T} - e^{(\theta + \beta)T} + 1 \right) \right)$. \hspace{1cm} (17)
By applying analogous argument with Equation (5), there are two distinct cases for finding the optimal $T^*$ are discussed as follows:

**Case 4.1** $\Delta_1 \geq \Delta_2$ (Building up inventory is profitable)

Using Appendix 1, the necessary conditions for maximizing $AP$ are $\partial AP / \partial T = 0$ and $\partial AP / \partial y = 0$. For the part $\partial AP / \partial T = 0$, we have

$$T = \frac{1}{\theta + \beta} \ln \left( \frac{B(\theta + \beta) + \alpha}{\alpha y(\theta + \beta) + \alpha} \right),$$

(18)

which indicates that $T$ is a function of $y$.

Substituting (18) into (16), we know that $AP$ is only a function of $y$. The first-order condition for finding the optimal $y^*$ is $dAP/dy = 0$, which leads to

$$-\frac{K(\theta + \beta)^2}{\Delta_1 - \Delta_2} = (\theta + \beta) (y - B) + [\alpha + y(\theta + \beta)] \ln \left( \frac{\alpha + B(\theta + \beta)}{\alpha + y(\theta + \beta)} \right),$$

(19)

To examine whether (19) has a unique solution, we set

$$H(y) = (\theta + \beta) (y - B) + [\alpha + y(\theta + \beta)] \ln \left( \frac{\alpha + B(\theta + \beta)}{\alpha + y(\theta + \beta)} \right).$$

(20)

Taking the first derivative of $H(y)$ with respect to $y$, we get

$$H'(y) = (\theta + \beta) \ln \left( \frac{B(\theta + \beta) + \alpha}{\alpha y(\theta + \beta) + \alpha} \right) > 0.$$  

(21)

By $H(0) = 0$ and (20), we know that $H(y)$ is negative and strictly increasing to zero at $y = B$. Consequently, we can obtain the following theorem.

**Theorem 1.** Under the condition $\Delta_1 \geq \Delta_2$, $I(0) = B$ and the following results state

If $H(0) \leq -K(\theta + \beta)^2 / (\Delta_1 - \Delta_2)$, then there exists a unique solution $y^*$ in (19) which maximizes $AP$ in (16).

If $H(0) > -K(\theta + \beta)^2 / (\Delta_1 - \Delta_2)$, then $y^* = 0$.

**Proof.** $AP$ is a continuous function of $y$ over the compact set $[0, B]$, and hence a maximum exists. The proof of part (a) immediately follows from (21) and $H(0) \leq -K(\theta + \beta)^2 / (\Delta_1 - \Delta_2) < H(B) = 0$. From Appendix 2, we show that $AP$ is a strictly concave function at $y^*$. Therefore, the unique optimal solution is an inner point if $H(0) < -K(\theta + \beta)^2 / (\Delta_1 - \Delta_2)$. Otherwise (i.e., $H(0) > -K(\theta + \beta)^2 / (\Delta_1 - \Delta_2)$), the optimal solution is at the boundary point $y = 0$ (Since $AP$ is zero at $y = B$, $y = B$ is not an optimal solution). The proof of part (b) is completed.

**Case 4.2** $\Delta_1 < \Delta_2$ (Building up inventory is not profitable)

The necessary conditions for maximizing $AP$ are $\partial AP / \partial T = 0$ and $\partial AP / \partial y = 0$. For the part $\partial AP / \partial T = 0$, we have
\[ K + (\Delta_1 - \Delta_2) \left( \frac{1}{\theta + \beta} \right) (y + \frac{\alpha}{\theta + \beta}) [(\theta + \beta)Te^{(\theta + \beta)T} - e^{(\theta + \beta)T} + 1] = 0. \] (22)

Taking the partial derivative of \( AP \) with respect to \( y \), we obtain

\[
\frac{\partial AP}{\partial y} = \frac{1}{T} \left[ (\Delta_1 - \Delta_2) \frac{1}{\theta + \beta} (e^{(\theta + \beta)T} - 1) \right] < 0. \] (23)

Therefore, we get \( y^* = 0 \). Substituting \( y^* = 0 \) into (22), we can get

\[
0 < \frac{-K(\theta + \beta)^2}{\alpha(\Delta_1 - \Delta_2)} = (\theta + \beta)Te^{(\theta + \beta)T} - e^{(\theta + \beta)T} + 1. \] (24)

Again, to examine whether (24) has a solution or not, we set

\[
G(T) = (\theta + \beta)Te^{(\theta + \beta)T} - e^{(\theta + \beta)T} + 1. \] (25)

Taking the first derivative of \( G(T) \) with respect to \( T \), we have

\[
G'(T) = (\theta + \beta)^2 Te^{(\theta + \beta)T} > 0. \] (26)

Since \( G(0) = 0 \), there exists a unique solution \( T \) (which is greater than 0) for (24). This is done in the following theorem.

**Theorem 2.** If \( \Delta_1 < \Delta_2 \), then the optimal ordering point \( y^* = 0 \), and there exists a unique solution \( T^* \) in (24) which maximizes \( AP \) in (16).

**Proof.** \( AP \) is a continuous function of \( T \) over the compact set \([0, T]\), and hence a maximum exists. Since \( AP \) is zero at \( T = 0 \), the optimal \( T^* \) is an inner point. From Appendix 3, we know that \( AP \) is a strictly concave function at \( T^* \). Thus, the unique solution to (24) is the optimal solution that maximizes \( AP \) in (16).

**Algorithm**

It is apparent from Theorem 1 and 2 that the value of \( AP \) is influenced by the values of \( \Delta_1 \) and \( \Delta_2 \). Consequently, the algorithm for determining the optimal cycle time \( T^* \), optimal ordering point \( y^* \) and optimal economic order quantity \( Q^* \) is summarized as follows:

**Step 1.** If \( \Delta_1 \geq \Delta_2 \) and \( H(0) \leq -K(\theta + \beta)^2 / (\Delta_1 - \Delta_2) \), then the optimal ordering point \( y^* \) can be determined by (19), the optimal cycle time \( T^* \) can be obtained by substituting \( y^* \) into (18), and the optimal economic order quantity \( Q^* = B - y^* \).

**Step 2.** If \( \Delta_1 \geq \Delta_2 \) and \( H(0) > -K(\theta + \beta)^2 / (\Delta_1 - \Delta_2) \), then the optimal ordering point \( y^* = 0 \), and thus the optimal cycle time \( T^* \) can be obtained by substituting \( y^* = 0 \) into (18), and the optimal economic order quantity \( Q^* = B \).

**Step 3.** If \( \Delta_1 < \Delta_2 \), then the optimal \( y^* = 0 \). By solving (24), we get the value for \( T \). Substituting \( y^* = 0 \) and \( T \) into (2) to find \( I(0) \). If \( I(0) < B \), then the optimal economic order quantity \( Q^* = I(0) \) and the optimal cycle time \( T^* = T \). Otherwise, \( Q^* = B \) and the optimal cycle time \( T^* \) can be determined by \( I(0) = B \).
Numerical examples

The numerical examples are given here to demonstrate the applicability of the proposed model.

Example 4.1 Let $K = $100 per cycle, $\alpha = 100$ units per unit time, $h = $1.0 per unit per unit time, $s = $0 per unit, $p = $6 per unit, $\theta = 0.2$ and $B = 250$. If $\beta = 0.05$, 0.10, 0.15 and 0.20, then $\Delta_1 < \Delta_2$. Using the Step 3 of the proposed algorithm, we can obtain the optimal solution that is the optimal ordering point $y^* = 0$, $T^*$ and $Q^*$. If $\beta = 0.25$ and 0.30, then $\Delta_1 \geq \Delta_2$ and $H(0) > -K(\theta + \beta)^2 / (\Delta_1 - \Delta_2)$. We can use Step 2 of the proposed algorithm and find the optimal solution that is the optimal ordering point $y^* = 0$, optimal economic order quantity $Q^* = B$ and $T^*$. If $\beta = 0.4$, 0.5, 0.6 and 0.7, then $\Delta_1 \geq \Delta_2$ and $H(0) \leq -K(\theta + \beta)^2 / (\Delta_1 - \Delta_2)$. From Step 1 of the proposed algorithm, the optimal solutions of $y^*$, $T^*$ and $Q^*$ can be attained. The computational results for the optimal values of $y$, $T$, $Q$ and $AP$ with respect to different values of $\beta$ are shown in Table 4.1.

<table>
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<tr>
<th>$\beta$</th>
<th>$y^*$</th>
<th>$Q^*$</th>
<th>$T^*$</th>
<th>$AP^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0</td>
<td>153.6248</td>
<td>1.300090</td>
<td>354.0565</td>
</tr>
<tr>
<td>0.10</td>
<td>0</td>
<td>182.7822</td>
<td>1.457292</td>
<td>372.0525</td>
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<td>1.717078</td>
<td>394.0700</td>
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<td>1.732868</td>
<td>420.1574</td>
</tr>
<tr>
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<td>0</td>
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<td>1.675048</td>
<td>445.7724</td>
</tr>
<tr>
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<td>0</td>
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</tr>
<tr>
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<tr>
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<td>649.2866</td>
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<tr>
<td>0.70</td>
<td>92.5158</td>
<td>157.4842</td>
<td>0.620301</td>
<td>719.6862</td>
</tr>
</tbody>
</table>

Table 4.1 reveals that (1) If $\Delta_1 < \Delta_2$, then the values of $Q^*$, $T^*$ and $AP^*$ increase when the value of $\beta$ increases. It implies that the increase of demand rate causes the increases of optimal economic order quantity, cycle time and average profit when building up inventory is not profitable. (2) If $\Delta_1 \geq \Delta_2$, then the values of $y^*$ and $AP^*$ increase but the values of $Q^*$ and $T^*$ decrease when the value of $\beta$ increase. It shows that a higher demand rate causes higher values of optimal ordering point and average profit, but lower values of economic order quantity and cycle time.

5. CONCLUSION

This article presents the inventory models for deterioration items when the demand is a function of the selling price and stock on display. We also impose a limited
maximum amount of stock displayed in a supermarket without leaving a negative impression on customers. Under these conditions, a proposed model has been shown for maximizing profits. Then, the properties of the optimal solution are discussed as well as its solution algorithm and numerical examples are presented to illustrate the model. In addition, in order to reflect an important fact that most firms have no pricing power in today’s business competition, we study a special case that the selling price is considered by predetermination. We then provide Theorems 1 and 2 to show the characteristics of the optimal solution and establish an easy-to-use algorithm to determine the optimal cycle time, economic order quantity and ordering point. Furthermore, we discover some intuitively reasonable managerial results. For example, if the benefit received from a unit of inventory is larger than the total cost per unit inventory, then the building up inventory is profitable and thus the beginning inventory should reach to the maximum allowable level. Otherwise, building up inventory is not profitable and the ending inventory should be zero. Finally, numerical examples are provided to demonstrate the applicability of the proposed model. The results also indicate that the effect of stock dependent selling rate on the system behavior is significant, and hence should not be ignored in developing the inventory models. The sensitivity analysis shows the influence effects of parameters on decision variables.

The proposed models can further be enriched by incorporating inflation, quantity discount, and trade credits etc. Besides, it is interested to extend the proposed model to multi-item inventory systems based on limited shelf space or to consider the demand rate which is a polynomial form of on-hand inventory dependent demand. Finally, we may extend the deterministic demand function to stochastic fluctuating demand patterns.

APPENDIX

Appendix 1. If $\Delta_1 \geq \Delta_2$, then $AP$ is an increasing function of $T$.

To prove $\frac{\partial AP}{\partial T} > 0$, we set

$$f(x) = xe^x - e^x + 1, \text{ for } x \geq 0. \quad (A.1)$$

Then (A.1) yields $f'(x) = xe^x > 0$. So, $f(x)$ is an increasing function of $x$ for $x \geq 0$. We get

$$f(x) > f(0) = 0. \quad (A.2)$$

Let $x = (\theta + \beta)T$. Using (A.1) and (A.2), we obtain

$$(\theta + \beta)Te^{(\theta + \beta)T} - e^{(\theta + \beta)T} + 1 > 0, \text{ for } T > 0. \quad (A.3)$$

Applying (5) and (A.3), we have $\frac{\partial AP}{\partial T} > 0$. 
Appendix 2. If $\Delta_1 \geq \Delta_2$, then $AP$ is strictly concave at $y^*$.

From (19), we know the second-order derivative of $AP$ with respect to $y$ as:

$$\frac{\partial^2 AP}{\partial y^2} = \frac{1}{T} \left( (\Delta_1 - \Delta_2) \left( \frac{1}{\theta + \beta} \right) \left[ \frac{\alpha + B(\theta + \beta)}{\alpha + y(\theta + \beta)} - 1 \right] \left( \frac{-1}{\alpha + y(\theta + \beta)} \right) \right) < 0,$$

(A.4)

which implies $AP$ is strictly concave at $y^*$.

Appendix 3. If $\Delta_1 < \Delta_2$, then $AP$ is strictly concave at $T^*$.

Applying (22) and $y^* = 0$, we obtain the second-order derivative:

$$\frac{\partial^2 AP}{\partial T^2} = (\Delta_1 - \Delta_2) \alpha T e^{\theta T} \beta T < 0,$$

(A.5)

which implies $AP$ is strictly concave at $T^*$.

REFERENCES


