STRICT BENSON PROPER-$\varepsilon$-EFFICIENCY IN VECTOR OPTIMIZATION WITH SET-VALUED MAPS

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Abstract: In this paper the notion of Strict Benson proper-$\varepsilon$-efficient solution for a vector optimization problem with set-valued maps is introduced. The scalarization theorems and $\varepsilon$-Lagrangian multiplier theorems are established under the assumption of ic-cone-convexlikeness of set-valued maps.

Keywords: Ic-cone-convexlikeness, Set-valued Maps, strict Benson proper-$\varepsilon$-efficiency, scalarization, $\varepsilon$-Lagrangian Multipliers.

MSC: 90C26, 90C29, 90C30, 90C46.

1. INTRODUCTION

In the study of vector optimization, the theory of efficiency plays an important role. Kuhn and Tucker [8] and later Geoffrion [6] observed that certain efficient points exhibit some abnormal properties and to eliminate such anomalous solutions in large set of efficient solutions, they introduced the concept of proper efficiency. Borwein [2] and Benson [1] proposed proper efficiency for vector

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The purpose of this paper is to introduce the notion of Strict Benson proper-\(\varepsilon\)-efficient solution for vector optimization problems with set-valued maps as a generalization of Benson proper efficient solution [1]. We study the relationship of Strict Benson proper-\(\varepsilon\)-efficient solution with $\varepsilon$-strict efficient solution given by Li et al. [10]. An alternative theorem is presented in section 3 for ic-cone-convexlikeness set-valued maps, which were introduced by Sach [13], and scalarization theorems and $\varepsilon$-Lagrangian Multiplier theorems are established in sections 4 and 5.

2. DEFINITIONS AND NOTATIONS

Let \(X\) be locally convex topological vector space and \(Y, Z\) be real locally convex Hausdorff topological vector spaces; let \(D \subset Y, E \subset Z\) be pointed closed convex cones. For a set \(A \subset Y\), we write cone \(A = \{\lambda a : \lambda \geq 0, a \in A\}\).

The closure and interior of the set \(A\) are denoted by \(\text{cl} A\) and \(\text{int} A\). A convex subset \(B\) of cone \(A\) is a base of \(A\) if \(0 < \text{cl} B\) and \(A = \text{cone} B\). Let \(Y^*\) be the dual space of \(Y\), the positive dual cone \(D^*\) of \(D\) is defined as \(D^* = \{f \in Y^* : f(y) \geq 0 \text{ for all } y \in D\}\). The set \(D^*\) of strictly positive functions is defined as \(D^* = \{f \in Y^* : f(y) > 0 \text{ for all } y \in D \setminus \{0\}\}\).

For a set-valued map \(F : X \to 2^Y\) the domain of \(F\), denoted by \(\text{dom} F\), is defined as \(\text{dom} F = \{x \in X : F(x) \neq \emptyset\}\), and the image of \(F\), denoted as \(\text{im} F\), is defined as \(\text{im} F = F(X) = \bigcup_{x \in X} F(x)\).

Benson [1] introduced the following definition of proper efficiency.

**Definition 2.1.** If \(S\) is non empty set in \(Y\) and \(D\) is a convex cone in \(Y\), then \(y \in S\) is called Benson proper efficient point of \(S\) over \(D\) written as \(y \in \text{BPMin}[S, D]\) if \(\text{clcone}(S + D - y) \cap (-D) = \emptyset\) \hspace{1cm} (1)

Now we introduce the notion of Strict Benson proper-\(\varepsilon\)-efficient point of a set \(S\) over a cone \(D\).

**Definition 2.2.** Let \(S\) be a non empty set in \(Y\), \(D\) be a convex cone in \(Y\) and \(\varepsilon \in D\), then \(\bar{y} \in S\) is called a Strict Benson proper-\(\varepsilon\)-efficient point of \(S\) over \(D\) written as \(\bar{y} \in \text{BP-}\varepsilon\text{-Min}[S, D]\) if \(\text{clcone}(S + \varepsilon - \bar{y}) \cap (-D \setminus \{0\}) = \emptyset\) \hspace{1cm} (2)
It is easy to verify that \( BPMin[S, D] \subset BP-e-Min[S, D] \).

The following example illustrates the proper containment, that is, there exists \( \bar{y} \notin BP-e-Min[S, D] \) but \( \bar{y} \in BPMin[S, D] \).

**Example 2.3.** Let \( Y = \mathbb{R}^2 \), \( D = \{(x, y) : x \leq y, y \geq 0\} \), \( e = \left(\frac{3}{7}, \frac{2}{3}\right) \),
\[
S = \left\{ \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{7}, 1\right), \left(\frac{3}{8}, \frac{3}{7}\right), (1, 1) \right\}, \bar{y} = (1, 1), \text{then } S + e - \bar{y} = \left\{ \left(1, \frac{2}{7}\right), \left(\frac{2}{5}, \frac{3}{7}\right), (3, 3), \left(\frac{3}{4}, \frac{3}{7}\right) \right\}
\]
and \( \text{clcone}(S + e - \bar{y}) \cap (-D \setminus \{0\}) = \emptyset \).

Thus, \( \bar{y} \notin BP-e-Min[S, D] \).

Also, \( S - g = \left\{ \left(-\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{2}, 0\right), \left(\frac{3}{8}, \frac{3}{7}\right), (0, 0) \right\} \) which shows that \( \left(-\frac{1}{2}, \frac{1}{2}\right) \in \text{clcone}(S + D - g) \cap (-D \setminus \{0\}) \) and \( \text{clcone}(S + D - g) \cap (-D \setminus \{0\}) \neq \emptyset \).

Thus, \( \bar{y} \notin BP-e-Min[S, D] \).

**Definition 2.4.** Let \( S \) be a non empty subset of \( Y \), \( D \) be a convex cone in \( Y \), \( B \) be a base of \( D \), \( e \in D \), then \( \bar{y} \in S \) is called an \( e \)-strictly minimal \( \epsilon \)-efficient point which is defined as follows.

It is shown in the following theorem that every \( e \)-strict minimal \( \epsilon \)-efficient point of \( S \) with respect to \( B \) is a base of \( D \).

**Theorem 2.5.** \( \epsilon \)-Fmin\[S, B\] \( \subset \) \( BP-e-Min[S, D] \).

**Proof.** Let \( \bar{y} \in \epsilon \)-Fmin\[S, B\] which implies that there is a neighborhood \( U \) of 0 such that \( \text{clcone}(S + e - \bar{y}) \cap (U - B) = \emptyset \).

Now, to show \( \bar{y} \in BP-e-Min[S, D] \), we have to prove \( \text{clcone}(S + e - \bar{y}) \cap (-D \setminus \{0\}) = \emptyset \).

On the contrary, suppose that there exists \( y' \in \text{clcone}(S + e - \bar{y}) \cap (-D \setminus \{0\}) \). It follows that \( y' \in \text{clcone}(S + e - \bar{y}) \) and \( y' 
\in \text{clcone}(S + e - g) \). Thus, \( y' \in \text{clcone}(S + e - \bar{y}) \cap (-D \setminus \{0\}) \) which gives that \( y' = -d \), for \( d \in D \setminus \{0\} \). Since \( B \) is a base of \( D \), therefore \( D = \text{cone } B \), which gives that \( d = \lambda b \), for \( \lambda \geq 0, b \in B \). It follows that \( y' = -d = -\lambda b \).

Clearly, \( \frac{y'}{\lambda} = -b \in (U - B) \) and also, \( \frac{y'}{\lambda} \in \text{clcone}(S + e - g) \). Thus, \( \frac{y'}{\lambda} \in \text{clcone}(S + e - \bar{y}) \cap (U - B) \), which gives a contradiction.

**Remark 2.6.** The following example illustrates that the set of Strict Benson proper-\( e \)-efficient points is not contained in the set of \( e \)-strictly minimal efficient points.

**Example 2.7.** Let \( Y = \mathbb{R}^2 \), \( S = \left\{ \left(-\frac{1}{2}, \frac{1}{2}\right), \left(-2, \frac{2}{3}\right), \left(\frac{3}{10}, \frac{3}{7}\right) \right\}, D = \{(x, y) : x \leq 0, y \leq 0\}, \( B = \{(x, y) : x + y + 1 = 0, x \leq 0, y \leq 0\} \) be a base of \( D \), \( e = \left(-\frac{1}{2}, 0\right), \bar{y} = \left(-\frac{1}{2}, \frac{1}{2}\right) \).
We consider the following set-valued optimization problem:  

\[
\text{minimize } \{ x, y : x^2 + y^2 < \frac{1}{4} \}, \text{ subject to } S + \varepsilon - \bar{y} = \left\{ \left( \frac{1}{2}, 0 \right), \left( -2, \frac{1}{4} \right), \left( \frac{1}{10}, 1 \right) \right\}
\]

and \( \text{clcone}(S + \varepsilon - \bar{y}) \cap (-D \setminus \{0\}) = \emptyset \), which gives that \( \bar{y} \in \text{BP}-\varepsilon-\text{Min}[S, D] \).

Also, \( \left( \frac{1}{2}, 1 \right) \in \text{clcone}(S + \varepsilon - \bar{y}) \cap (U - B), \) which shows that \( \bar{y} \notin \varepsilon-\text{Fmin}[S, B] \). Thus \( \text{BP}-\varepsilon-\text{Min}[S, D] \notin \varepsilon-\text{Fmin}[S, B] \).

### 3. THEOREM OF ALTERNATIVE

A theorem of the alternative will be established in this section for ic-D-convexlike set-valued maps, which were introduced by Sach [13] and are defined as follows.

**Definition 3.1.** A set-valued map \( F : X \rightarrow 2^Y \) is called ic-D-convexlike on \( X \) if \( \text{intcone}(F(X) + D) \) is a convex cone and \( F(X) + D \subset \text{clintcone}(F(X) + D) \).

**Theorem 3.2.** Let \( \text{int} D \neq \emptyset \) and let the set-valued map \( F : X \rightarrow 2^Y \) be ic-D-convexlike on \( X \) then, one and only one of the following statements is true:

(I) there exists \( x \in X \) such that \( F(x) \cap (-\text{int} D) \neq \emptyset \)

(II) there exists \( \mu \in (D^* \setminus \{0\}) \) such that \( \mu(y) \geq 0 \), for all \( y \in F(X) \).

**Proof.** Assume that both (I) and (II) hold. Then there exist \( x \in X, y \in F(x) \) such that \( y \in -\text{int} D \), which gives that \( \mu(y) < 0 \) for every \( \mu \in D^* \setminus \{0\} \). This contradicts (II). Thus (I) and (II) cannot hold simultaneously.

Now, we show that if (I) is not true, then (II) holds. Suppose that \( F(X) \cap (-\text{int} D) = \emptyset \).

Now we claim that \( \text{intcone}(F(X)+D) \cap (-\text{int} D) = \emptyset \). Indeed, let \( y' \in \text{intcone}(F(X)+D) \cap (-\text{int} D) \), then there exist \( x \in X, d \in D, \lambda > 0 \) such that \( y' = \lambda(F(x)+d) \in \text{int} D \), which gives that \( \frac{\lambda}{\lambda} - d \in F(x) \). Since \( y' \in -\text{int} D \), \( \lambda > 0 \) therefore, \( \frac{\lambda}{\lambda} = -d \in \text{int} D \), which implies that \( \frac{\lambda}{\lambda} - d \in \text{int} D \). Thus, \( \frac{\lambda}{\lambda} - d \in F(X) \cap (-\text{int} D) \), which contradicts (I).

By the assumption \( F \) is ic-D-convexlike on \( X \), we have that \( \text{intcone}(F(X)+D) \) is a convex cone. Thus, by separation theorem for convex sets in topological vector spaces as given by Jahn [7], there exists \( \mu \in Y^* \setminus \{0\} \) such that \( \mu(y + t\bar{d}) \geq 0 \) for all \( y \in F(X), d \in D \) and \( t > 0 \).

We assert that \( \mu(d) \geq 0 \) for all \( d \in D \) otherwise, suppose that there exists \( \bar{d} \in D \) with \( \mu(\bar{d}) < 0 \). Then we will have \( \mu(y + t\bar{d}) = \mu(y) + t\mu(\bar{d}) < 0 \), for given \( y \) and sufficiently large \( t \), which contradicts (5). Thus, \( \mu \in D^* \setminus \{0\} \). Letting \( t \to 0 \) in (5), we obtain \( \mu(y) \geq 0 \) for all \( y \in F(X) \). This implies that (II) is true.

### 4. SCALARIZATION

We consider the following set-valued optimization problem:

\[
\text{(VP)} \quad \begin{array}{l}
\text{minimize } F(x) \\
\text{subject to } G(x) \cap (-E) \neq \emptyset
\end{array}
\]
where $X_0 \subset X$ is a nonempty set, $E \subset Z$ is a pointed closed convex cone, $-E = \{-x : x \in E\}$, $F : X_0 \to 2^Y$, $G : X_0 \to 2^Z$ are set-valued maps. The set of feasible solutions of (VP) is denoted by $V = \{x \in X_0 : G(x) \cap (-E) \neq \emptyset\}$.

We now introduce Strict Benson proper-$\varepsilon$-efficient solution of (VP).

**Definition 4.1.** A point $x \in V$ is said to be Strict Benson proper-$\varepsilon$-efficient solution of (VP) if $F(x) \cap \text{BP-$\varepsilon$-Min}[F(V), D] \neq \emptyset$.

A pair $(x, y)$ is said to be Strict Benson proper-$\varepsilon$-minimizer of (VP) if $y \in F(x) \cap \text{BP-$\varepsilon$-Min}[F(V), D]$.

Corresponding to the set-valued optimization problem (VP), we associate the following scalar optimization problem:

$$\text{(SP)} \mu \underset{x \in V}{\text{min}} (\mu F)(x)$$

where $\mu \in D^+ \setminus \{0\}$

**Definition 4.2.** Let $x \in V$, $y \in F(x)$, then $x$ is said to be an $\varepsilon$-minimal solution of (SP)$\mu$, if $\mu(y) \leq \mu(y) + \mu(\varepsilon)$ for all $y \in F(V)$ and $(x, y)$ is said to be an $\varepsilon$-minimizer pair of (SP)$\mu$.

The fundamental results characterizing Strict Benson proper-$\varepsilon$-minimizer of (VP) in terms of $\varepsilon$-minimizer of (SP)$\mu$ are now discussed.

**Theorem 4.3.** Let $\mu \in D^\#$ be fixed. If $(x, y)$ is an $\varepsilon$-minimizer pair of (SP)$\mu$, then $(x, y)$ is a Strict Benson proper-$\varepsilon$-minimizer pair of (VP).

**Proof.** Since $(x, y)$ is an $\varepsilon$-minimizer of (SP)$\mu$, therefore

$$\mu(y) \leq \mu(y) + \mu(\varepsilon), \text{ for all } y \in F(V). \quad (6)$$

Now, we shall show that $(x, y)$ is a Strict Benson proper-$\varepsilon$-minimizer pair of (VP).

It is enough to show that, $\text{clcone}(F(V) + \varepsilon - y) \cap (-D \setminus \{0\}) = \emptyset$. Indeed, if there exists $y' \in \text{clcone}(F(V) + \varepsilon - y) \cap (-D \setminus \{0\})$, then, there exist $\{y_n\} \subset F(V)$ and $\{\lambda_n\} \subset \mathbb{R}_+$ such that $y' = \lim_{n \to \infty} \lambda_n(y_n + \varepsilon - y)$ and $y' \in -D \setminus \{0\}$

By using (6), we have $\mu(y') = \lim_{n \to \infty} \lambda_n \mu(y_n + \varepsilon - y) \geq 0 \quad (7)$

Since $\mu \in D^\#$ and $y' \in -D \setminus \{0\}$, therefore $\mu(y') < 0$, which contradicts (7).

Thus, we conclude $(x, y)$ is a Strict Benson proper-$\varepsilon$-minimizer pair of (VP).

Below we give an example to illustrate the above theorem.

**Example 4.4.** Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $Z = \mathbb{R}^2$ and $D = \{(x, y) : x \geq y, y \geq 0\}$, $E = \{(x, y) : y \geq x, x \leq 0\}$ and $\varepsilon = \left(\frac{1}{2}, \frac{1}{2}\right)$. 

...
Define $F: X \to 2^Y$, as $F(x) = \begin{cases} [0, 1] \times [0, 1] & \text{if } x \geq 0 \\ [0, x^2] \times [0, x^2] & \text{if } x < 0 \end{cases}$ and define $G : X \to 2^Z$, as
\[
G(x) = \begin{cases} ([0, x], \frac{1}{2}) & \text{if } x \geq 0 \\ [-2, -1] \times [-2, -1] & \text{if } x < 0 \end{cases}
\]
The feasible set of the problem (VP) is $V = \{x : x \geq 0\}$.
Let $\tilde{L} = \{\bar{y} = (\frac{1}{2}, 0) \in F(x)\}$.
Then, $F(V) = [0, 1] \times [0, 1]$, $\mu(\tilde{y}) = \frac{1}{2}$, $\mu(\bar{y}) = \frac{1}{2}$ and $\mu(\tilde{y}) \leq \mu(y) + \mu(\bar{y})$, for all $y \in F(V)$, which implies that, $(\tilde{x}, \tilde{y})$ is an $\epsilon$-minimizer pair of (SP). Since $\epsilon - \tilde{y} = (0, \frac{1}{2})$, therefore $\operatorname{clcone}(F(V) + \epsilon - \tilde{y}) \cap (-D \setminus \{0\}) = \emptyset$. Thus, $(\tilde{x}, \tilde{y})$ is a Strict Benson proper-$\epsilon$-minimizer pair of (VP).

**Theorem 4.5.** Let $F : X \to 2^Y$ be defined as $F(x) = F(x) + \epsilon - \bar{g}$ for all $x \in X$ and $\bar{F}$ be ic-D-convexlike on $V$. If $(\tilde{x}, \tilde{g})$ is a Strict Benson proper-$\epsilon$-minimizer pair of (VP), then there exists $\mu \in D^* \setminus \{0\}$ such that $(\tilde{x}, \tilde{g})$ is an $\epsilon$-minimizer pair of (SP). $\mu$.

**Proof.** Let $(\tilde{x}, \tilde{g})$ be a strict Benson proper-$\epsilon$-minimizer pair of (VP). Then $\tilde{x} \in V$ and $\tilde{y} \in F(x) \cap \operatorname{BP-\epsilon-Min}[F(V), D]$, which gives that $\operatorname{clcone}(F(V) + \epsilon - \tilde{y}) \cap (-D \setminus \{0\}) = \emptyset$. It follows that $(F(V) + \epsilon - \tilde{y}) \cap (-\operatorname{int} D) = \emptyset$.

By assumption $\bar{F}$ is ic-D-convexlike on $V$, then by Theorem 3.1 there exists $\mu \in D^* \setminus \{0\}$ such that $\mu(z) \geq 0$ for all $z \in \bar{F}(V)$, which gives that $\mu(y + \epsilon - \tilde{y}) \geq 0$, for all $y \in F(V)$. Thus, $\mu(y) + \mu(\epsilon) \leq \mu(\bar{y})$, for all $y \in F(V)$. Hence, $(\tilde{x}, \tilde{y})$ is an $\epsilon$-minimizer pair of (SP). $\mu$.

### 5. $\epsilon$-LAGRANGIAN MULTIPLIER THEOREMS

In this section we present two $\epsilon$-Lagrangian Multiplier theorems which show that a Strict Benson proper-$\epsilon$-minimizer of the constrained set-valued vector optimization problem (VP) is exactly a Strict Benson proper-$\epsilon$-minimizer for an appropriate unconstrained set-valued vector optimization problem under certain conditions.

Let $L(Z, Y)$ be the space of continuous linear operators from $Z$ to $Y$, and let $L_r(Z, Y) = \{T \in L(Z, Y) : T(E) \subset D\}$.

Denote by $(F, G)$ the set-valued map from $X$ to $Y \times Z$ defined by $(F, G)(x) = F(x) \times G(x)$, for all $x \in X$.

If $\mu \in Y^*$, $T \in L(Z, Y)$, we define $\mu F : X \to 2^R$ and $F + TG : X \to 2^Y$ as $(\mu F)(x) = \mu(F(x))$ and $(F + TG)(x) = F(x) + T(G(x))$, respectively.

The set-valued Lagrange map of (VP), $L : X_0 \times L_r(Z, Y) \to 2^\mathcal{X}$ is defined as
\[
L(x, T) = F(x) + T(G(x)), \quad \text{where } (x, T) \in X_0 \times L_r(Z, Y).
\]

We consider the following unconstrained set-valued minimization problem associated with (VP) for a fixed $T \in L_r(Z, Y)$
\[
(VP)_T \quad \underset{x \in X_0}{\operatorname{D-min}} L(x, T)
\]
Sach [13] gave the following result for ic-cone convexlike set-valued maps.

**Lemma 5.1.** Let intcone(imF + D) ≠ φ, then F is an ic-D-convexlike if and only if k intcone(imF + D) + (1 - k) cone(imF + D) ⊂ intcone(imF + D), for all k ∈ (0, 1).

We establish the following result by using the above lemma.

**Lemma 5.2.** If (F, G) is ic-D-convexlike on X and intcone(im(F, G) + D × E) ≠ φ then for μ ∈ D* \ {0}, (μF, G) is ic-(R × E)-convexlike on X.

**Proof.** Let (F, G) be ic-D-convexlike on X. Then by using Lemma 5.1 k(intcone(imF + D), intcone(imG + E)) + (1 - k)(cone(imF + D), cone(imG + E)) ⊂ (intcone(imF + D), intcone(imG + E)), for all k ∈ (0, 1) which gives that k intcone(imF + D) + (1 - k) cone(imF + D) ⊂ intcone(imF + D), for all k ∈ (0, 1) and k intcone(imG + E) + (1 - k) cone(imG + E) ⊂ intcone(imG + E), for all k ∈ (0, 1).

Now, it is enough to show k intcone(im μF + R × ) + (1 - k) cone(im μF + R × ) ⊂ intcone(im μF + R × ), for all k ∈ (0, 1).

Let y' ∈ k intcone(im μF + R × ) + (1 - k) cone(im μF + R × ), then there exists λ1 > 0, λ2 ≥ 0, x1, x2 ∈ X, y1 ∈ F(x1), y2 ∈ F(x2) and r1, r2 ∈ R × such that y' = kλ1(μ(y1) + r1)+(1-k)λ2(μ(y2)+r2), which gives that y' = μ(λ1 y1 + (1-k)λ2 y2) + kλ1 r1 + (1-k)λ2 r2. Now kλ1 y1 + (1-k)λ2 y2 ∈ k λ1(F(x1) + D) + (1-k)λ2(F(x2) + D) ⊂ k intcone(imF + D) + (1-k) cone(imF + D) ⊂ intcone(imF + D), for all k ∈ (0, 1).

This gives that there exists λ3 > 0, x3 ∈ X, y3 ∈ F(x3) and d3 ∈ D such that kλ1 y1 + (1-k)λ2 y2 = λ3(y3 + d3).

Then, y' = μ(λ3(y3 + d3) + kλ1 r1 + (1-k)λ2 r2) = λ3μ(y3) + λ3μ(d3) + kλ1 r1 + (1-k)λ2 r2 ⊂ intcone(im μF + R ×), as μ(d3) ∈ R × and (kλ1 r1 + (1-k)λ2 r2)/λ3 ∈ R ×. Thus, (μF, G) is ic-(R × E)-convexlike on X.

We now give ε-Lagrangian multiplier theorems:

**Theorem 5.3.** Let Y be locally convex space, D be closed convex pointed cone with a non empty interior. Let F : X → 2 Y be defined as F(x) = F(x) + ε - g for all x ∈ X, F be ic-D-convexlike on V and (F, G) be ic-(D × E)-convexlike on X₀ and intcone(im(F, G) + D × E) ≠ φ. Further, let (VP) satisfy the generalized Slater constraint qualification, that is, imG ∩ (-int E) ≠ φ. If (x, g) is a Strict Benson proper-ε-minimizer of (VP) and 0 ∈ T(G(x)), then there exist T ∈ Lε(Z,Y) and μ ∈ D* \ {0} such that (x, g) is an ε-minimizer pair of the following scalar set-valued optimization problem (VP)ε : \[
\min_{x \in X} \mu(F(x) + T(G(x))
\]

If μ ∈ D* then (x, g) is a Strict Benson proper-ε-minimizer of (VP)T.

**Proof.** Since (x, g) is a Strict Benson proper-ε-minimizer of (VP), therefore by Theorem 4.2 there exists μ ∈ D* \ {0} such that
\[ \mu(F(x) + \varepsilon - \bar{y}) \geq 0, \text{ for all } x \in V \] 

(8)

Let us define \( H : X_0 \to 2^{\mathbb{R}^n} \) as \( H(x) = \mu(F(x) + \varepsilon - \bar{y}) \times G(x) = (\mu F)(x) + (\mu \varepsilon - \mu(\bar{y})) \), since \((F,G)\) is ic-(\(D \times E\))-convexlike on \( X_0 \), and \( \text{intcone}(\text{im} F, G + D \times E) \neq \emptyset \), therefore by Lemma 5.2 \( H \) is ic-(\(\mathbb{R}^n \times E\)) convexlike on \( X_0 \).

Further, (8) implies that the system \( x \in X_0, H(x) \cap (-\text{int}(\mathbb{R}^n \times E)) \neq \emptyset \) has no solution. Hence by Theorem 3.1, there exists \((\lambda, \psi) \in \mathbb{R}^n \times E' \setminus [0,0], y \in F(x), z \in G(x) \) such that \( \lambda \mu(y + \varepsilon - \bar{y}) + \psi(z) \geq 0 \) for all \( x \in X_0 \)

(9)

We claim that \( \lambda \neq 0 \).

On the contrary, suppose that \( \lambda = 0 \) then, we have \( \psi \in E' \setminus [0] \). By generalized slater constraint qualification, there exists \( x_1 \in X_0 \) such that \( G(x_1) \cap (-\text{int} E) \neq \emptyset \). Thus, there exists \( z_1 \in G(x_1) \) such that \( z_1 \in (-\text{int} E) \), which gives that \( \psi(z_1) < 0 \) but on substituting \( \lambda = 0 \) and taking \( x = x_1 \) and \( z = z_1 \) in (9), we have \( \psi(z_1) \geq 0 \), which is a contradiction. Hence \( \lambda > 0 \).

Since \( \mu \in D^* \setminus [0] \). We can choose \( d \in D \setminus [0] \) such that \( \lambda \mu(d) = 1 \).

We define the operator \( T : Z \to Y \) as \( T(z) = \psi(z)d \)

(10)

then \( T \in Z_+(Z,Y) \) and \( 0 \in \psi(G(x))d = T(G(x)) \).

Hence, \( g \in F(x) + T(G(x)) \).

From (9) and (10), we obtain

\[ \lambda \mu(y + \varepsilon + T(z)) = \lambda \mu(y) + \lambda \mu(\varepsilon) + \psi(z) \lambda \mu(d) = \lambda \mu(y) + \lambda \mu(\varepsilon) + \psi(z) \geq \lambda \mu(y), \text{ for all } x \in X_0 \text{ which gives that} \]

\[ \mu(\bar{y}) \leq \mu(y + T(z)) + \mu(\varepsilon) \text{ for all } x \in X_0, y \in F(x) \text{ and } z \in G(x). \]

Hence, \((\bar{x}, \bar{y})\) is an \( \varepsilon \)-minimizer pair of set-valued optimization problem \((\overline{VP})_{\mu}\).

If \( \mu \in D^* \), then by using Theorem 4.1, we get that \((\bar{x}, \bar{y})\) is Strict Benson proper-\(\varepsilon\)-minimizer of \((\overline{VP})_{\tau}\).

We now establish the converse of Theorem 5.1.

**Theorem 5.4.** Let \( \bar{x} \in V, \bar{y} \in F(\bar{x}) \). If there exists \( T \in L_+(Z,y) \) such that \( 0 \in T(G(x)) \), and \((\bar{x}, \bar{y})\) is a Strict Benson proper-\(\varepsilon\)-minimizer \((\overline{VP})_{\tau}\), then \((\bar{x}, \bar{y})\) is a Strict Benson proper-\(\varepsilon\)-minimizer of \((VP)\).

**Proof.** Since \( 0 \in T(G(x)) \), and \((\bar{x}, \bar{y})\) is a Strict Benson proper-\(\varepsilon\)-minimizer of \((\overline{VP})_{\tau}\), therefore, \( \bar{y} \in F(\bar{x}) + T(G(\bar{x})) \) and \( \text{clcone}(F(V) + T(G(V)) + \varepsilon - \bar{y}) \cap (-D \setminus [0]) = \emptyset \).

(11)

Now we shall show that \((\bar{x}, \bar{y})\) is a Strict Benson proper-\(\varepsilon\)-minimizer pair of \((VP)\).

For that, it is enough to show that \( \text{cone}(F(V) + \varepsilon - \bar{y}) \cap (-D \setminus [0]) = \emptyset \).

On the contrary, if \( y' \in \text{cone}(F(V) + \varepsilon - \bar{y}) \cap (-D \setminus [0]) \) then there exists \( x \in V, y \in F(x), k > 0 \) such that \( y' = k(y + \varepsilon - \bar{y}) \) and \( y' \in (-D \setminus [0]) \).

Since \( 0 \in T(G(x)) \), therefore, \( y' \in \text{clcone}(F(V) + T(G(V)) + \varepsilon - \bar{y}) \), which contradicts (11).

Hence, \((\bar{x}, \bar{y})\) is a Strict Benson proper-\(\varepsilon\)-minimizer of \((VP)\).
6. CONCLUSION

The objective of this paper is to introduce the notion of Strict Benson proper-$\varepsilon$-efficient solution for vector optimization problem with set-valued maps to generalize the notion of Benson proper efficiency and establish an alternative theorem. We also obtain scalarization theorems and $\varepsilon$-Lagrangian multiplier theorems under the assumption of ic-cone-convexlikeness.

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