REDISTRIBUTION OF THE INFLUENCES IN SYSTEMS WITH SEMI-RIGID JOINTS ON ELASTIC FOUNDATIONS

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Abstract. Most often, in the case of typefied prefabricated systems, foundations are designed and constructed as prefabricated elements as well. When the structure is exposed to high intensity loading and founded on a weak soil, beam foundations are often used instead of pad foundations. Beam foundations, stiffening girders, as well as beams which support the façade elements or partition walls, are treated as beams on elastic foundations, while joint of these girders to pad foundations or vertical support elements of a precast structure can be treated as semi-rigid. Modeling of systems with semi-rigid joints on elastic foundations, for different levels of rigidity of connections, is illustrated by an example of a frame under static loading. On the basis of the results of the calculation carried out in this paper, it is evident that taking into account the elastic foundations of the member and the corresponding degree of fixation of the member on this foundations significantly affects the magnitude of the forces in the cross-sections of the member, and on the redistribution of influences in the entire structure. Yet, the foundations-and-soil interaction exerted the highest influence on the magnitude of the stress in the foundation structure itself.

Key words: precast structure, semi-rigid connection, redistribution of influences.

1. INTRODUCTION

The systems where the mutual joints of the members are not absolutely rigid, but allow a certain degree of relative displacement in the directions of all generalized displacements are considered the systems with semi-rigid joints. It has been observed that the degree of rigidity of a joint is particularly important in prefabricated structures, because even a small degree of fixation of a joint which has been designed as flexible favorably affects redistribution of influences, while insufficiently secured and presumed rigid joints can have negative consequences during distribution of stresses in the structure.
In typical prefabricated systems which are produced in large batches, for the purpose of quality control, and avoiding dependence on the weather conditions, and in order to reduce the cost of structures, most frequently the foundations are designed and constructed as prefabricated elements. When the structural loads are of considerable intensity, or the foundation soil bearing capacity is low, frequently instead of foundation footings, the foundation beams are constructed. In essence, they represent the members which are apart from the external load also exposed to the influences of the soil resistance and they are considered beams on elastic foundations. The stiffening beams, as well as the beams bearing the façade elements, partition walls and the similar can also be considered beams on elastic foundations, and the joints of these beams with the foundation footings or vertical bearing elements in prefabricated buildings can be considered semi-rigid.

2. Members with semi-rigid joints in nodes

2.1 Interpolation functions for a member with semi-rigid joints in nodes

The relation of the displacement of the arbitrary point of the member axis and displacement parameters at the ends of the member, in the case of straight member stressed on bending in the plane, can most simply be obtained starting from the homogenous differential equation of bending:

$$\frac{d^4v(x)}{dx^4} = 0,$$

Whose solution can be presented in the form of a polynomial of the third order:

$$v(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3.$$  \hspace{1cm} (2)

The function $v(x)$ at unit generalized displacement $q_m=1 \ (m=1,2,3,4)$, whereby all other generalized displacements are $q_n=0, \ n \neq m$, is known as the interpolation function.

When the node $i$ of the member that is semi-rigid on both ends is imparted a unit translational movement $q_i=1$, while all other generalized displacement are equal to zero, according to the Fig. 1(a) it can be written as:

$$\alpha_{ik}^* = \left[ \frac{1}{\xi_{ik}} - (1 - \mu_{ik}^*) \frac{b_{ik}}{a_{ik}} \right] \frac{1}{\xi_i}, \quad \alpha_{ki}^* = \left[ \frac{1}{\xi_{ki}} - (1 - \mu_{ki}^*) \frac{b_{ki}}{a_{ki}} \right] \frac{1}{\xi_i}$$  \hspace{1cm} (3)

where $\mu_{ik}, \mu_{ki}$ are degrees of rigidity of joints at the ends of the member, and which can be determined experimentally or through calculation as $\mu_{ik}^* = \frac{\phi_{ik}^*}{\phi_i}, \ \mu_{ki}^* = \frac{\phi_{ki}^*}{\phi_i}, \ a \ \phi_i, \phi_k^*$ are angles of rotation of the nodes, while $\phi_{ik}^*, \phi_{ki}^*$ are the angles of rotation of cross-sections at the ends of the members.
Interpolation functions for the member with semi-rigid joints at the ends \( i \) and \( k \), are determined starting from the function \( v(x) \) and the corresponding boundary conditions at the ends of the member in the cases \( q_{m}=1 \) \((m=1,2,3,4)\), as displayed in [4], have the following forms:

\[
\begin{align*}
N_{1}^{*}(x) &= 1 - \left(1 - \alpha_{ik}^{*}\right)x - \frac{2\alpha_{ik}^{*} + \alpha_{ki}^{*}}{l}x^{2} + \frac{\alpha_{ik}^{*} + \alpha_{ki}^{*}}{l^{2}}x^{3} \\
N_{2}^{*}(x) &= \mu_{ik}x - \frac{2\mu_{ik} - \mu_{ki} + \alpha_{ki}^{*}}{l}x^{2} + \frac{\mu_{ik} - \mu_{ki} + \alpha_{ki}^{*}}{l^{2}}x^{3} \\
N_{3}^{*}(x) &= \left(1 - \alpha_{ik}^{*}\right)x + \frac{2\alpha_{ik}^{*} + \alpha_{ki}^{*}}{l}x^{2} - \frac{\alpha_{ik}^{*} + \alpha_{ki}^{*}}{l^{2}}x^{3} \\
N_{4}^{*}(x) &= \left(\mu_{ik} - \mu_{ki}\right)x - \frac{2\mu_{ik} + \mu_{ki} - 2\alpha_{ki}^{*}}{l}x^{2} + \frac{\mu_{ik} + \mu_{ki} - \alpha_{ki}^{*}}{l^{2}}x^{3}
\end{align*}
\]

The interpolation function \( N_{m}^{*}(x) \), the so called, function of form, represents the elastic line of a semi-rigidly fixed member at both ends, due to the generalized displacement \( q_{m}=1 \), \( m = 1, 2, 3, 4 \), while all other generalized displacements are \( q_{n}=0 \), \( n \neq m \). Interpolation functions given by the terms (4) represent the Hermite's polynomials of the first order, and their diagrams are displayed in the Fig. 1. In the boundary cases, when the member is rigidly fixed at the ends \( i \) and \( k \) \( (\mu_{ik} = \mu_{ki} = 1) \) or when it is rigid at one end and with the joint on the other \((\mu_{ik} = 1, \mu_{ki} = 0)\) the terms (4) assume their well known values [3].
2.2 Rigidity matrix of the member with semi-rigid joints in nodes

The rigidity matrix of a member with semi-rigid joints, exposed to bending in the plane has the following form:

\[
\mathbf{k}^* = \begin{bmatrix}
k_{11}^* & k_{12}^* & k_{13}^* & k_{14}^* \\
k_{22}^* & k_{23}^* & k_{24}^* \\
k_{33}^* & k_{34}^* \\
k_{44}^*
\end{bmatrix}_{\text{sim}}.
\]  

The elements of this matrix are determined with the aid of derivations of other interpolation functions, such as defined by the term (6), and they are in the function of the joints rigidity degree:

\[
k_{11}^* = \frac{4EI}{l} \left[ \alpha_{ik}^2 + \alpha_{ik} \alpha_{ki}^* + \alpha_{ki}^2 \right],
\]

\[
k_{12}^* = \frac{2EI}{l} \left[ 2(\alpha_{ik}^* \mu_{ik} + \alpha_{ki}^2 \ell - \alpha_{ki} \mu_{ik}) - \alpha_{ik} \mu_{ki} + \alpha_{ik}^* \alpha_{ki}^* \ell + \alpha_{ki}^* \mu_{ik} \right],
\]

\[
k_{13}^* = -\frac{4EI}{l} \left[ \alpha_{ik}^2 + \alpha_{ik} \alpha_{ki}^* + \alpha_{ki}^2 \right] = -k_{11}^*.
\]

Other elements of the rigidity matrix are given in the paper [4].

2.3 Vector of equivalent load for the member with semi-rigid joints in nodes

The vector of equivalent load is determined in the following manner:

\[
\mathbf{Q}^T = \int_0^l p(x) N^*(x) \, dx.
\]  

For the uniformly distributed load \( p(x) = p = \text{const} \), on the basis of (4) and (8), the vector of equivalent load is obtained in the following form:
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\[ Q^* = \frac{pl^2}{12} \left( \frac{6}{I} + \alpha_{ik} - \alpha_{ki} \right) \]

\[ = \frac{pl^2}{2} \left( \frac{1}{6} \right) \] for \( \mu_k = \mu_j = 1 \)

\[ = \frac{pl^2}{2} \left( \frac{1}{6} \right) \] for \( \mu_k = 1, \mu_j = 0 \)

In the boundary cases when the member is fixed at the ends \( i \) and \( k \) or when it is fixed at one end and with a flexible joint on the other \( (\mu_k = 1, \mu_j = 0) \), the elements of the vector of equivalent load obtain well known values (9).

3 MODELS OF THE BEAM ON ELASTIC FOUNDATIONS

Here will be considered beams on elastic foundations behaving in accordance to the Winkler's hypothesis, [1] where the resistance of the base, as a reactive transversal load, is proportional to transversal displacements of any point of the beam:

\[ p_r(x) = cv(x), \] (10)

where:

- \( p_r \) - reactive transversal load,
- \( c \) - proportionality coefficient \([\text{kN/m}^2]\).

Elastic foundations can be modeled by the series of elastic spring which are evenly arranged along the beam whose rigidity is \( c \).

The differential equation of the beam on the elastic beam in this case can be obtained directly on the basis of the differential equation of the member exposed to bending, when the transversal load \( p(x) \) substituted by the load \( p(x) - cv(x) \):

\[ \frac{d^4v(x)}{dx^4} + \frac{c}{EI} v = \frac{p(x)}{EI}. \] (11)

The correct solution of the differential equation (11) in the closed form contains within the trigonometric and hyperbolic functions, so with its aid it is difficult to describe various boundary conditions and arbitrary loads which can occur in the beams on the elastic foundations. Because of it, here will briefly be presented application of discrete models for calculation of beams on the elastic foundations.

Discretization of the beam with smaller elements facilitates modeling of various boundary conditions and the arbitrary load along the axis of the member and is suitable for matrix formulation of structural design. Due to their simplicity, universality, clear formulation as well as suitability for programming, the deformation method will be implemented.

Discrete models for the analysis of beams on the elastic foundations can, according to [1] be classified in two groups:
1. The model with springs concentrated in the nodes of the member,
2. The model with springs arranged continuously along the length of the member.

Fig. 2 Models of a beam on elastic foundations with springs

3.1 The model with springs concentrated in the nodes of the member

In this model, the influence of the elastic foundations is transmitted by the springs concentrated in the nodes of the member, Fig. 2 (c).

The rigidity of these springs is:

\[ k_1 = k_2 = \frac{E I}{2} \]  \hspace{1cm} (12)

This discrete model utilizes the well known conventional rigidity matrix of the member fully fixed on both ends, provided that in the direction of transversal displacement 1 and 3, on the diagonal positions, the rigidity values of the springs \( k_1 \) and \( k_2 \), are added, so the rigidity matrix assumes the form:

\[
\begin{bmatrix}
  k_{11} & k_{12} & k_{13} & k_{14} \\
  k_{21} & k_{22} & k_{23} & k_{24} \\
  k_{31} & k_{32} & k_{33} & k_{34} \\
  k_{41} & k_{42} & k_{43} & k_{44}
\end{bmatrix}
= l
\begin{bmatrix}
  \frac{12}{l^3} + k_1 & 6 & -\frac{12}{l^3} & 6 \\
  4 & \frac{6}{l^2} & \frac{2}{l} & 0 \\
  -\frac{12}{l^3} + k_2 & -\frac{6}{l^2} & \frac{6}{l^2} & 6 \\
  \frac{4}{l^3} & \frac{6}{l^2} & \frac{2}{l} & \frac{4}{l}
\end{bmatrix}
\]  \hspace{1cm} (13)

For the semi-rigidly connected members, instead of the conventional rigidity matrix, the matrix (5) can be used.

When using this model, the further calculation procedure is in the classical linear beams. However, one should bear in mind that this solution is approximate, and that for the more accurate solutions it is necessary to discreticize the member with a higher number of elements which significantly increases the time of calculation.

In the case when the beam is elevated from the foundations, the model can be easily adapted by assuming the values of \( k_1 \) and \( k_2 \) are zero.

By comparison of experimental and analytically obtained results, Bowles proposes that for the end springs, twice as high values are adopted for \( k_1 \) and \( k_2 \).
3.2 The model with springs arranged continuously along the length of the member

In the Fig. 2(b) the model of the beam on the elastic foundations with springs continuously arranged along the length of the member.

The starting point for determination of the rigidity matrix on the elastic foundations according to [3] are functions of form (4), Fig. 1. If \( v(x) \) is presented in the usual way with the products of interpolation functions and parameters of displacement \( (q) \) the term (10) becomes:

\[
p_r(x) = c N q.
\]

On the basis of the virtual displacement principle, the continuously distributed transversal load along the axis of the member \( p_r(x) \) is equivalent to the load on the ends of the member \( R_m, m=1, 2, 3, 4, \) which, taking into consideration (14) can be presented in the form:

\[
R = \int_0^l \nabla^T p_r \, dx = c \int_0^l \nabla^T N q \, dx = C q,
\]

Where:

\[
C = c \int_0^l \nabla^T N \, dx,
\]

is the rigidity matrix of the elastic foundations. The elements of this matrix are determined on the basis of [1] according to the term:

\[
C_{mn} = c \int_0^l N_m(x) N_n(x) \, dx, \quad m,n = 1,...,4.
\]

In the case of the member which is semi-rigidly connected in the nodes, the elements of the rigidity matrix of the elastic foundations can be determined on the basis of the term:

\[
C_{mn} = c \int_0^l N_m^*(x) N_n^*(x) \, dx, \quad m,n = 1,...,4.
\]

On the basis of (4) in (18) the elements of the rigidity matrix of the elastic foundations are obtained as the function of the degree of rigidity of joints on the ends of the members:

\[
C_1 = \frac{cl}{420} \left[ 140 + (42 \alpha_{sl} - 28 \alpha_{sl}) l + (4 \alpha_{sl}^2 - 6 \alpha_{sl} \alpha_{sl}^2) l^2 \right],
\]

\[
C_2 = \frac{cl}{420} \left[ 4(\mu_{sl} \mu_{sl} + \alpha_{sl}^2 l^2) + 6(\mu_{sl} \mu_{sl} - \mu_{sl} \alpha_{sl}^2 l - 8 \mu_{sl} \alpha_{sl}^2 l^2) \right],
\]

\[
C_3 = \frac{cl}{420} \left[ 140 - (28 \alpha_{sl} - 42 \alpha_{sl}) l + (4 \alpha_{sl}^2 + 6 \alpha_{sl} \alpha_{sl}^2) l^2 \right],
\]

\[
C_4 = \frac{cl}{420} \left[ 4(\mu_{sl} \mu_{sl} + \alpha_{sl}^2 l^2) + 6(- \mu_{sl} \mu_{sl} + \mu_{sl} \alpha_{sl}^2 l - 8 \mu_{sl} \alpha_{sl}^2 l^2) \right].
\]
And the matrix of rigidity of the elastic foundations has the form:

\[
C^* = \begin{bmatrix}
C_{11}^* & C_{12}^* & C_{13}^* & C_{14}^* \\
C_{12}^* & C_{22}^* & C_{23}^* & C_{24}^* \\
C_{13}^* & C_{23}^* & C_{33}^* & C_{34}^* \\
C_{14}^* & C_{24}^* & C_{34}^* & C_{44}^*
\end{bmatrix}
\] (20)

In the boundary cases when \( \mu_{ik} = \mu_{kj} = 1 \), as well as \( \mu_{ik} = 1, \mu_{kj} = 0 \), the terms (19) have known values given in [1].

In the case when the beam is elevated from the foundations, the matrix \( \bar{C}^* \) is zero.

The rigidity matrix of the semi-rigidly connected member which is supported by the elastic foundations is equal to the sum of the rigidity matrix and the semi-rigidly connected member (5) and the matrix of the rigidity of foundations (20), i.e.

\[
k^* = (k^* + \bar{C}^*).
\] (21)

The vector of the equivalent load is determined in the same way as in the member loaded to bending, as presented in chapter 2.3 of this paper.

In this model too, having in mind the approximate character of the procedure, the accuracy depends on the degree of discretization of beam on the elastic foundations, but here the satisfactory accuracy is achieved with considerably lower number of elements.

3.3 An example of the frame on elastic foundations

For the purpose of illustrating the degree of accuracy of certain models, V. Simončič, having analyzed the beam of the elastic foundations with a high number of elements per model with the springs concentrated in the nodes, as well as the model with continuously arranged springs along the length of the member, demonstrated the advantage of using the model with continuously distributed springs, because with a relatively small number of elements the solution very quickly converges towards the accurate solution [1]. If the beam is approximated with the aid of one element, the error is only 6.78%. Therefore, for
analysis in this paper the model of the frame on elastic foundations with continuously distributed springs is chosen.

For the purpose of determination of the influences of the semi-rigidly connected member supported by the elastic foundations, the calculation for the reinforced concrete frame [1] has been conducted. In the Fig. 4 the structure of this frame has been displayed, along with its mathematical model and the discretization was adopted. The calculation has been performed by the matrix analysis method, whereby the degree of fixation of joints at the ends of the members 2 and 3 has been varied. The member 3 has been treated as the beam on the elastic foundations, so the rigidity matrix of this member has been determined according to (21). The rigidity matrix of the member 2 has been determined on the basis of (5) because in the node 1 it has been semi-rigidly connected, while for the member 1 which was rigidly connected at the ends, the conventional matrix of rigidity on the basis of [3] has been adopted. In the Fig. 5 the values of the bending moments at the ends of the members for various degrees of fixation on the ends of the member 3 have been displayed.

Fig. 4 Model of the frame on elastic foundations with continuously distributed springs

4. CONCLUSION

Using the rigidity matrix of the semi-rigidly connected member which is supported by the elastic foundations, the members which are treated as the beams of the elastic foundations can be included into the calculation as members with semi-rigid joints, which to a significant extent increases the generality of the calculation, and it is particularly important in computer aided structural design.
On the basis of the results of the calculation carried out in this paper, it is evident that taking into account the elastic foundations of the member and the corresponding degree of fixation of the member on this foundations significantly affects the magnitude of the forces in the cross-sections of the member, and on the redistribution of influences in the entire structure. Yet, the foundations-and-soil interaction exerted the highest influence on the magnitude of the stress in the foundation structure itself.

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