A THEOREM ON A REPRESENTATION OF
∗–REGULARLY VARYING SEQUENCES

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Abstract. In this paper we shall prove a theorem on a representation of ∗–regularly varying sequences in the sense of Karamata [1].

1. Introduction and results

Consider the sequences \((c_n)\), \(c_n > 0\) \((n \in \mathbb{N})\) which are nondecreasing and satisfy the following asymptotic condition

\[
\lim_{\lambda \to 1+} \lim_{n \to +\infty} \frac{c_{[n\lambda]}}{c_n} = 1.
\]

Such sequences are called ∗–regularly varying, and they have an important role in the analysis of divergent sequential processes (see e.g. [6] ).

Condition (1) is weaker than the Karamata condition of regular variability and stronger than the condition of \(O\)–regular variability (see e.g. [2], [5], [8] and [9] ).

Relation (1) obviously means that for any such sequence, the function

\[
k_0(\lambda) = \lim_{n \to +\infty} \frac{c_{[n\lambda]}}{c_n}
\]

is right continuous at \(\lambda = 1\).

We shall first define an important class of functions.

CRV is the class of all measurable functions \(F: [a, +\infty) \mapsto (0, +\infty)\) \((a > 0)\) such that \(F(x(t)) \sim F(y(t))\) as \(t \to +\infty\), for any two functions \(x, y\) with the properties

\[
\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} y(t) = +\infty,
\]

and \(x(t) \sim y(t)\) as \(t \to +\infty\).

This class is investigated in detail in the papers [3] and [4].

Proposition 1. If \((c_n)\) is an arbitrary nondecreasing sequence of positive numbers, then the following assertions are equivalent:

\[
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\]
(a) \( c_{[x]} (x \geq 1) \) belongs to the class \( CRV \);
(b) \( (c_n) \) is a *-regularly varying sequence.

**Corollary 1.** The function \( k_0(\lambda) \) is defined for every \( \lambda > 0 \).

**Corollary 2.** For any *-regularly varying sequence \( (c_n) \) we have
\[
\lim_{\lambda \to 1} k_0(\lambda) = 1.
\]

**Corollary 3.** For any *-regularly varying sequence \( (c_n) \) and arbitrary positive \( s, t \) we have
\[
k_0(st) \leq k_0(s)k_0(t).
\]

**Corollary 4.** For an arbitrary *-regularly varying sequence \( (c_n) \), the function \( k_0(\lambda) \) is continuous in \( \lambda > 0 \).

**Corollary 5.** If \( (c_n) \) is an arbitrary nondecreasing sequence of positive numbers, then the following assertions are equivalent:
(a) \( (c_n) \) is a *-regularly varying sequence;
(b) For arbitrary mappings \( K_1, K_2 : \mathbb{N} \to \mathbb{N} \) with the properties
\[
\lim_{n \to +\infty} K_1(n) = \lim_{n \to +\infty} K_2(n) = +\infty
\]
and \( K_1(n) \sim K_2(n) \) as \( n \to +\infty \), one has \( c_{K_1(n)} \sim c_{K_2(n)} \) as \( n \to +\infty \);
(c) \[
\lim_{n \to +\infty} \frac{c_{[\lambda n]}}{c_n} = 1.
\]

The asymptotic condition (c) is in fact the Schmidt convergence condition (see e.g. [7] ).

The next theorem is a representation theorem for *-regularly varying sequences.

**Theorem 1.** Let \( (c_n) \) be an arbitrary nondecreasing sequence of positive numbers. Then the following assertions are equivalent:
(a) \( (c_n) \) is a *-regularly varying sequence;
(b) There is an \( n_0 \in \mathbb{N} \) such that
\[
c_n = \exp \left\{ \bar{\mu}_n + r_1(\log n) + \sum_{k=n_0}^{n} \frac{\delta_k}{k} \right\}
\]
for every \( n \geq n_0 \), where the sequence \( \bar{\mu}_n \to 0 \) as \( n \to +\infty \), \( r_1 \) is a bounded and uniformly continuous function on the interval \([\log n_0, +\infty)\), and \((\delta_n)\) is a bounded sequence.

**Remark.** It can be proved that all previous results remain true for sequences \((c_n)\) which are not necessarily nondecreasing if the condition \( \lambda \to 1^+ \) in (1) is replaced by \( \lambda \to 1 \).

2. **Proofs of results**

**Proof of Proposition 1.** By some results from [4,p.454] the implication (a) \( \implies \) (b) is trivial.

(b) \( \implies \) (a). Let \((c_n)\) be a \(\ast\)-regularly varying sequence. From (1) we find that

\[
\lim_{n \to +\infty} \frac{c[\lambda n]}{c_n} < +\infty
\]

for some \( a > 1 \) and all \( \lambda \in [1, a) \). Hence, the function \( k_0(\lambda) \) is finite for all \( \lambda \in [1, a) \). For an arbitrary fixed \( \lambda \in [1, a) \) define

\[
k(\lambda) = \lim_{x \to +\infty} \frac{c[\lambda x]}{c[x]}.
\]

Then

\[
k(\lambda) \leq \lim_{x \to +\infty} \frac{c[\lambda x]}{c[x]} \cdot \lim_{x \to +\infty} \frac{c[\lambda x]}{c[\lambda x]} \leq k_0(\lambda) \cdot \lim_{x \to +\infty} \frac{c[\lambda x]}{c[\lambda x]}.
\]

Since \((\lambda x)/[\lambda x]\) \( \to 1^+ \) as \( x \to +\infty \), we find that for a fixed \( \lambda \) and all \( x \geq x_0 \), \((\lambda x)/[\lambda x]\) \( \in [1, 1 + \delta] \). Hence,

\[
\lim_{x \to +\infty} \frac{c[\lambda x]}{c[\lambda x]} = \lim_{x \to +\infty} \frac{c[\lambda x]}{c[\lambda x]} \leq k_0(1 + \delta)
\]

for all \( \delta > 0 \). Thus for a given \( \lambda \) and all \( \delta \in (0, a-1) \) we have \( k(\lambda) \leq k_0(\lambda) \cdot k_0(1 + \delta) \). Since the function \( k_0 \) is right continuous at \( \lambda = 1 \), we have that \( k(\lambda) \leq k_0(\lambda) \) for a fixed \( \lambda \in [1, a) \). Therefore \( \lim_{\lambda \to 1^+} k(\lambda) \leq \lim_{\lambda \to 1^+} k_0(\lambda) \leq 1 \). Since \( \lambda \) is an nondecreasing function, we have that \( \lim_{\lambda \to 1^+} k(\lambda) = 1 \) and \( \lim_{\lambda \to 1^-} k(\lambda) \leq 1 \). Finally, since the function \( k(\lambda) \) is defined on an interval \([1 - a', 1 + a'']\) \((a', a'' > 0)\), we get that \( c[x] \) belongs to the class of \( O \)-regularly varying functions \((1))\), that is we have

\[
k(\lambda) = \lim_{x \to +\infty} \frac{c[\lambda x]}{c[x]} < +\infty
\]
for all $\lambda > 0$.

Hence we get $1 = k(1) \leq k(\lambda) \cdot k(1/\lambda)$, and consequently $k(1/\lambda) \to c \geq 1$ as $\lambda \to 1+$. This gives that $\lim_{\lambda \to 1^+} k(\lambda) = 1$, that is $\lim_{\lambda \to 1} k(\lambda) = 1$, which by some results from [4,p.454] yields that $c[x]$ belongs to the class $CRV$. \hfill $\Box$

**Remark.** Since $k_0(\lambda) \leq k(\lambda)$ for $\lambda > 0$, and $k(\lambda) \leq k_0(\lambda) \cdot k_0(1+\delta)$ for all $\delta > 0$, we find that $k_0(\lambda) = k(\lambda)$ for all $\lambda > 0$. Therefore, using the properties of the index function of $CRV$ functions, Corollaries 2, 3 and 4 follow immediately.

**Proof of Corollary 5.** (a) $\implies$ (b). If $(c_n)$ is a *-regularly varying sequence, then the function $F(x) = c[x]$ ($x \geq 1$) belongs to the class $CRV$.

Next, let $K_1$ and $K_2$ be arbitrary functions with the properties in (b). Then, it is easily seen that the functions $k_1, k_2$ defined by

$$k_i(x) = K_i(n) \quad (x \in [n, n+1); \; n \in \mathbb{N}; \; i = 1, 2)$$

have the properties $\lim_{x \to +\infty} k_1(x) = \lim_{x \to +\infty} k_2(x) = +\infty$, and $k_1(x) \sim k_2(x)$ as $x \to +\infty$. Hence $\lim_{x \to +\infty} (F(k_1(x))/F(k_2(x))) = 1$, and consequently $c_{K_1(n)} \sim c_{K_2(n)}$ as $n \to +\infty$.

(b) $\implies$ (c). Assume (b), and let $(\lambda_k)$ be an arbitrary sequence such that $\lim_{k \to +\infty} \lambda_k = 1$. Then

$$S_k = \frac{[\lambda_k k] + 1}{k} \geq \lambda_k \geq \frac{[\lambda_k k]}{k}$$

for every $k \in \mathbb{N}$, and $S_k, \frac{[\lambda_k k]}{k} \to 1$ as $k \to +\infty$. Putting $n_1(k) = [\lambda_k k]$, $n_2(k) = [\lambda_k k] + 1$, $n_3(k) = k$, we have that $n_1, n_2, n_3 : \mathbb{N} \to \mathbb{N}$, $n_1(k), n_2(k), n_3(k) \to +\infty$ as $k \to +\infty$ and

$$\lim_{k \to +\infty} \frac{n_1(k)}{n_3(k)} = \lim_{k \to +\infty} \frac{n_2(k)}{n_3(k)} = 1.$$

Since

$$1 = \lim_{k \to +\infty} \frac{c_{n_1(k)} n_3(k)}{c_{n_3(k)}} = \lim_{k \to +\infty} \frac{c_{[\lambda_k k]}}{c_k} \leq \lim_{k \to +\infty} \frac{c_{n_2(k)} n_3(k)}{c_{n_3(k)}} = 1,$$

we get condition (c).
(c) \implies (a). Assuming (c), and taking an arbitrary \( \epsilon > 0 \), we find some \( k_0 \in \mathbb{N} \) and a \( \delta > 0 \) such that \( 1 - \epsilon \leq c_{\lfloor \lambda k \rfloor}/c_k \leq 1 + \epsilon \) whenever \( k \geq k_0 \) and \( |1 - \lambda| < \delta \). Hence

\[
1 - \epsilon \leq \lim_{k \to +\infty} \frac{c_{\lfloor \lambda k \rfloor}}{c_k} \leq 1 + \epsilon,
\]

thus \( |k_0(\lambda) - 1| \leq \epsilon \) if \( |1 - \lambda| < \delta \). This means that

\[
\lim_{\lambda \to 1} \lim_{k \to +\infty} \frac{c_{\lfloor \lambda k \rfloor}}{c_k} = 1,
\]

so that we have (a). \( \square \)

**Proof of Theorem 1.** (a) \implies (b). Let \((c_n)\) be a \( \ast \)-regularly varying sequence. Then, by Proposition 1, the function \( F(x) = c_{\lfloor x \rfloor} \ (x \geq 1) \) belongs to the class \( CRV \). By [4] there is a \( B > 0 \) such that for any \( n \geq B \) one has

\[
c_n = F(n) = \exp \left\{ \tilde{\mu}(n) + r(\log n) + \int_B^n \frac{\epsilon(t)}{t} \, dt \right\},
\]

where the functions \( \epsilon(x) \) and \( \tilde{\mu}(x) \) are bounded measurable functions in \([B, +\infty)\), \( r(x) \) is a uniformly continuous bounded function in \([\log B, +\infty)\), and we have \( \lim_{x \to +\infty} \tilde{\mu}(x) = 0 \). Putting \( n_0 = \lfloor B \rfloor + 1 \) and \( s = \int_{B}^{n_0} \frac{\epsilon(t)}{t} \, dt \in \mathbb{R} \), we get that the function \( r_1(t) = r(t) + s \) is bounded and uniformly continuous in \( t \in [\log n_0, +\infty) \). Hence, for \( n \geq n_0 \) we have

\[
c_n = \exp \left\{ \tilde{\mu}_n + r_1(\log n) + \sum_{k=n_0}^{n} \frac{\delta_k}{k} \right\},
\]

where \( \lim_{n \to +\infty} \tilde{\mu}_n = 0 \), \( r_1 \) is a bounded and uniformly continuous function on the interval \([\log n_0, +\infty)\), \( \delta_k = k \int_{k-1}^{k} \frac{\epsilon(t)}{t} \, dt \) \((k \geq n_0 + 1)\) and \( \delta_{n_0} = 0 \). Finally, we find that

\[
|\delta_k| = k \cdot \left| \int_{k-1}^{k} \frac{\epsilon(t)}{t} \, dt \right| \leq k \cdot \sup_{t \geq k-1} |\epsilon(t)| \cdot \log \left( 1 + \frac{1}{k-1} \right) \leq 2 \cdot \sup_{t \geq k-1} |\epsilon(t)| < M,
\]

for any \( k \geq n_0 + 1 \), since the function \( \epsilon(t) \) is bounded on \([B, +\infty)\).

(b) \implies (a). Assuming (b), let \( \lambda > 1 \) and \( n \geq n_0 \). Then

\[
\frac{c_{\lfloor \lambda n \rfloor}}{c_n} = \exp \left\{ \tilde{\mu}_{\lfloor \lambda n \rfloor} - \tilde{\mu}_n + r_1(\log \lfloor \lambda n \rfloor) - r_1(\log n) + \sum_{k=n+1}^{[\lambda n]} \frac{\delta_k}{k} \right\},
\]

where \( \tilde{\mu}_{\lfloor \lambda n \rfloor} = \tilde{\mu}(\lfloor \lambda n \rfloor) \).
Since
\[ \lim_{\lambda \to 1^+} \lim_{n \to +\infty} \left( \tilde{\mu}_{[\lambda n]} - \tilde{\mu}_n \right) = 0, \]
\[ \left| \sum_{k=n+1}^{[\lambda n]} \frac{\delta_k}{k} \right| \leq \sup_{k \geq n+1} |\delta_k| \cdot \int_{n+1}^{[\lambda n]+1} \frac{dt}{t-1} = \sup_{k \geq n+1} |\delta_k| \cdot \log \left( \frac{\lambda n}{n} \right), \]
that is \( \lim_{\lambda \to 1^+} \lim_{n \to +\infty} \left| \sum_{k=n+1}^{[\lambda n]} \frac{\delta_k}{k} \right| = 0, \) and \( r_1 \) is a uniformly continuous function on the interval \([\log n_0, +\infty),\) we get that
\[ \lim_{\lambda \to 1^+} \lim_{n \to +\infty} \frac{c_{[\lambda n]}}{c_n} = 1. \]

In other words, \((c_n)\) is a \( \ast \)-regularly varying sequence. □

References


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