SOME NEW DIFFERENCE SEQUENCES SPACES
DEFINED BY AN ORLICZ FUNCTION

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Abstract. In this paper we introduce some new difference sequence spaces combining lacunary sequences and Orlicz functions. We establish some inclusion relations between these spaces.

1. Introduction

Let $\ell_\infty$ and $c$ denote the Banach spaces of real bounded and convergent sequences $x = ( x_i )$ normed by $\| x \| = \sup_i | x_i |$, respectively.

A sequence of positive integers $\theta = ( k_r )$ is called \"lacunary\" if $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by $\theta$ will be denoted by $I_r = ( k_{r-1}, k_r )$ and $q_r = k_r / k_{r-1}$. The space of lacunary strongly convergent sequence $N_\theta$ was defined by Freedman et al [5] as:

$$N_\theta = \{ x : \lim_{r \to \infty} h_r^{-1} \sum_{i \in I_r} | x_i - s | = 0, \text{ for some } s \}$$

An Orlicz function is a function $M : [0, 1) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of $M$ is replaced by subadditivity, then this function is called a modulus functions (see, Ruckle [13]).

Let $w$ be the spaces of all real or complex sequence $x = ( x_i )$. Lindenstrauss and Tzafriri [8] used the idea of Orlicz function to defined the following sequence spaces.

$$l_M = \{ x : \sum_{i=1}^\infty M \left( \frac{| x_i |}{\rho} \right) < \infty, \rho > 0 \}$$

which is called an Orlicz sequence spaces $l_M$ is a Banach space with the norm,

$$\| x \| = \inf \{ \rho > 0 : \sum_{i=1}^\infty M \left( \frac{| x_i |}{\rho} \right) \leq 1 \}.$$
Strongly almost convergent sequence was introduced and studied by Maddox [10] and also independently by Freedman et al [5].

Parashar and Chaudhary [12] have introduced and examined some properties of the sequence spaces defined by using an Orlicz function $M$, which generalized the well-known Orlicz sequence spaces $[c,1,p]$, $[c,1,p]_0$ and $[c,1,p]_1$. It may be noted here that the spaces of strongly summable sequences were discussed by Maddox [9].

Kizmaç [6] was defined the sequence spaces

\[ l_\infty(\Delta) = \{ x = (x_i) : \sup \Delta x_i < \infty \}, \]

\[ c(\Delta) = \{ x = (x_i) : \lim_i |\Delta x_i - s| = 0 \text{ for some } s \}, \]

\[ c_0 (\Delta) = \{ x = (x_i) : \lim_i |\Delta x_i| = 0 \}, \text{ where } \Delta x_i = (x_i - x_{i+1}). \]

Subsequently difference sequence spaces has been discussed in Bilgin [2], Ahmad and Mursaleen [1], Malkowsky and Parashar [11], Et and Başarır [3], Et and Çolak [4] and others. The purpose of this paper is to introduce and study a concept of lacunary $\Delta$-convergence using Orlicz function and to examine inclusion relations among new spaces in the same way that $c(\Delta)$ is related to $c$.

Now we introduce the following sequence spaces:

**Definition 1.1** Let $M$ be an Orlicz function and $p = (p_i)$ be any bounded sequence of strictly positive real numbers. We have

\[ w_0^\Delta(M,p) = \{ x : \lim_{r \to \infty} h_r^{-1} \sum_{i \in I_r} M(\frac{\Delta x_i}{\rho})^{p_i} = 0, \rho > 0 \} \]

\[ w^\Delta(M,p) = \{ x : \lim_{r \to \infty} h_r^{-1} \sum_{i \in I_r} M(\frac{\Delta x_i - s}{\rho})^{p_i} = 0 \text{ for some } s, \rho > 0 \} \]

\[ w_0^\infty(M,p) = \{ x : \sup_r h_r^{-1} \sum_{i \in I_r} M(\frac{\Delta x_i}{\rho})^{p_i} < \infty, \rho > 0 \}, \]

where for convenience, we put $M(\frac{\Delta x_i}{\rho})^{p_i}$ instead of $M(\Delta x_i)^{p_i}$. If $x \in w^\Delta(M,p)$, we say that $x$ is lacunary $\Delta$-convergence to $s$ with respect to the Orlicz function $M$.

When $M(x) = x$, then we write $w_0^\Delta(p)$, $w^\Delta(p)$ and $w_0^\infty(p)$ for the spaces $w_0^\Delta(M,p)$, $w^\Delta(M,p)$ and $w_0^\infty(M,p)$, respectively. If $p_i = 1$ for all $i$, then $w_0^\Delta(M,p)$, $w^\Delta(M,p)$ and $w_0^\infty(M,p)$ reduce to $w_0^\Delta$, $w^\Delta$ and $w_0^\infty(M)$, respectively.

The following inequality will be used throughout the paper:

\[ |a_i + b_i|^{p_i} \leq C(|a_i|^{p_i} + |b_i|^{p_i}) \]

where $a_i$ and $b_i$ are complex numbers, $C = \max(1,2H^{-1})$, and $H = \sup p_i < \infty$

**2. Inclusion theorems**

By using (1), it is easy to prove the following theorem.
Theorem 2.1. Let $M$ be an Orlicz function and $p = (p_i)$ be a bounded sequence of strictly positive real numbers. Then $w^0(M, p)_\Delta$, $w^\theta(M, p)_\Delta$ and $w^\infty(M, p)_\Delta$ are linear spaces over the set of complex numbers.

Theorem 2.2. Let $M$ be an Orlicz function. If $\sup_i (M(x))^{p_i} < \infty$ for all fixed $x > 0$ then

$$w^\theta(M, p)_\Delta \subset w^\infty(M, p)_\Delta.$$ 

Proof. Let $x \in w^\theta(M, p)_\Delta$. There exists some positive $\rho_1$ such that

$$\lim_{r \to \infty} h_r^{-1} \sum_{i \in I_r} M\left(\frac{|\Delta x_i|}{\rho_1}\right)^{p_i} = 0.$$ 

Define $\rho = 2\rho_1$. Since $M$ is non-decreasing and convex, by using (1.1), we have

$$\sup_r h_r^{-1} \sum_{i \in I_r} M\left(\frac{|\Delta x_i - s|}{\rho}\right)^{p_i} = \sup_r h_r^{-1} \sum_{i \in I_r} M\left(\frac{|\Delta x_i|}{\rho_1}\right)^{p_i}$$

$$\leq C \left\{ \sup_r h_r^{-1} \frac{1}{2} M\left(\frac{|\Delta x_i - s|}{\rho_1}\right)^{p_i} + \sup_r h_r^{-1} \sum_{i \in I_r} \frac{1}{2} M\left(\frac{|s|}{\rho_1}\right)^{p_i} \right\}$$

$$< C \left\{ \sup_r h_r^{-1} \sum_{i \in I_r} M\left(\frac{|\Delta x_i - s|}{\rho_1}\right)^{p_i} + \sup_r h_r^{-1} \sum_{i \in I_r} M\left(\frac{|s|}{\rho_1}\right)^{p_i} \right\} < \infty.$$ 

Hence $x \in w^\infty(M, p)_\Delta$. This completes the proof.

Theorem 2.3. Let $M$ be an Orlicz function and $0 < h = \inf p_i$. Then $w^\infty(M, p)_\Delta \subset w^\theta(p)_\Delta$ if and only if

$$\lim_{r \to \infty} h_r^{-1} \sum_{i \in I_r} M(t)^{p_i} = \infty$$

for some $t > 0$.

Proof. Let $w^\infty(M, p)_\Delta \subset w^\theta(p)_\Delta$. Suppose that (2) does not hold. Therefore there are a subinterval $I_{(m)}$ of the set of interval $I_r$ and a number $t_0 > 0$, where $t_0 = \frac{|\Delta x_i|}{\rho}$ for all $i$, such that

$$h_r^{-1} \sum_{i \in I_{(m)}} M(t_0)^{p_i} \leq K < \infty, m = 1, 2, 3, ...$$

Let us define $x = (x_i)$ as following

$$\Delta x_i = \begin{cases} \rho t_0 &; i \in I_{(m)} \\ 0 &; i \notin I_{(m)} \end{cases}$$

Thus by (3), $x \in w^\infty(M, p)_\Delta$. But $x \notin w^\theta(p)_\Delta$. Hence (2) must hold.

Conversely, suppose that (2) holds and that $x \in w^\infty(M, p)_\Delta$. Then, for each $r$
Suppose that $x \notin w_0^\beta(p)\Delta$. Then, for some number $1 > \varepsilon > 0$, there is a number $i_0$ such that, for a subinterval $I_{r_1}$ of the set of interval $I_r$, $\frac{|\Delta x_i|}{\rho} > \varepsilon$ for $i \geq i_0$. From properties of the Orlicz function, we can write

$$M\left(\frac{|\Delta x_i|}{\rho}\right)^{p_i} \geq M(\varepsilon)^{p_i}$$

which contradicts (2), by using (4). Hence we get $w_\infty^\beta(M,p)\Delta \subset w_0^\beta(p)\Delta$. This completes the proof.

**Definition 2.1** An Orlicz function $M$ is said to satisfy the $\Delta_2$-condition for all values of $\mu$, if there exists a constant $L > 0$ such that $M(2\mu) \leq LM(\mu)$, $\mu \geq 0$.

It is also easy to see that always $L > 2$. The $\Delta_2$-condition equivalent to the satisfaction of inequality $M(T\mu) \leq LT M(\mu)$ for all values of $\mu$ and for all $T > 1$ (see, Krasnoselskii and Rutitsky [7]).

**Theorem 2.4** Let $0 < h = \inf p_i \leq p_i \leq \sup p_i = H < \infty$. For an Orlicz function $M$ which satisfies $\Delta_2$-condition, we have $w_0^\beta(M,p)\Delta \subset w_0^\beta(M,p)\Delta$ and $w_\infty^\beta(p)\Delta \subset w_\infty^\beta(M,p)\Delta$.

**Proof.** Let $x \in w_0^\beta(p)\Delta$. Then we have

$$h_r^{-1} \sum_{i \in I_r} \left(\frac{|\Delta x_i - s|}{\rho}\right)^{p_i} \rightarrow 0 \text{ as } r \rightarrow \infty,$$

for some $s$.

Let $\varepsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \leq t \leq \delta$. We can write

$$h_r^{-1} \sum_{i \in I_r} M\left(\frac{|\Delta x_i - s|}{\rho}\right)^{p_i} = h_r^{-1} \sum_{\substack{i \in I_r \\text{ s.t.} \\frac{|\Delta x_i - s|}{\rho} \leq \delta}} M\left(\frac{|\Delta x_i - s|}{\rho}\right)^{p_i} + h_r^{-1} \sum_{\substack{i \in I_r \\text{ s.t.} \\frac{|\Delta x_i - s|}{\rho} > \delta}} M\left(\frac{|\Delta x_i - s|}{\rho}\right)^{p_i}$$

For the first summation above, we immediately write

$$h_r^{-1} \sum_{\substack{i \in I_r \\text{ s.t.} \\frac{|\Delta x_i - s|}{\rho} \leq \delta}} M\left(\frac{|\Delta x_i - s|}{\rho}\right)^{p_i} \leq \max (\varepsilon, \varepsilon^h)$$

by using continuity of $M$. For the second summation, we will make following procedure. We have

$$\left(\frac{|\Delta x_i - s|}{\rho}\right)^{p_i} < 1 + \left(\frac{|\Delta x_i - s|}{\rho}\right)^{p_i} / \delta.$$
Since $M$ is non-decreasing and convex, it follows that

$$M \left( \frac{|\Delta x_i - s|}{\rho} \right) < M\{1 + \left( \frac{|\Delta x_i - s|}{\rho} \right) / \delta \} \leq \frac{1}{2} M(2) + \frac{1}{2} M\{2 \left( \frac{|\Delta x_i - s|}{\rho} \right) / \delta \}$$

Since $M$ satisfies $\Delta_2$-condition, we can write

$$M \left( \frac{|\Delta x_i - s|}{\rho} \right) \leq \frac{1}{2} L\{\left( \frac{|\Delta x_i - s|}{\rho} \right) / \delta \} M(2) + \frac{1}{2} L\{\left( \frac{|\Delta x_i - s|}{\rho} \right) / \delta \} M(2)$$

In this way, we write

$$h^{r-1}_r \sum_{i \in I_r} M \left( \frac{|\Delta x_i - s|}{\rho} \right)^{p_i} \leq \max(\varepsilon, \varepsilon^h) +$$

$$+ \max \{1, |LM(2) / \delta|^H \} h^{r-1}_r \sum_{i \in I_r} \left( \frac{|\Delta x_i - s|}{\rho} \right)^{p_i}$$

Taking the limit as $\varepsilon \to 0$ and $r \to \infty$, it follows that $x \in w^\theta(M, p) \Delta$. Following similar arguments we can prove that $w^\theta_0(p) \Delta \subset w^\theta(M, p) \Delta$ and $w^\theta_\infty(p) \Delta \subset w^\theta(M, p) \Delta$.

After step of this section, different inclusion relations among these sequence spaces are going to be studied. Now we have

**Theorem 2.5.** Let $M$ be an Orlicz function. Then the following statements are equivalent.

i) $w^\theta_0(p) \Delta \subset w^\theta_\infty(M, p) \Delta$

ii) $w^\theta_0(p) \Delta \subset w^\theta_\infty(M, p) \Delta$

iii) $\sup_r h^{r-1}_r \sum_{i \in I_r} M(t)^{p_i} < \infty$ for all $t > 0$.

**Proof.** i) $\Rightarrow$ ii): Let (i) holds. To verify (ii), it is enough to prove $w^\theta_0(p) \Delta \subset w^\theta_\infty(p) \Delta$. Let $x \in w^\theta_0(p) \Delta$. Then, there exist $r \geq r_0$, for $\varepsilon > 0$, such that

$$h^{r-1}_r \sum_{i \in I_r} \left( \frac{|\Delta x_i|}{\rho} \right)^{p_i} < \varepsilon.$$

Hence there exists $K > 0$ such that

$$\sup_r h^{r-1}_r \sum_{i \in I_r} \left( \frac{|\Delta x_i|}{\rho} \right)^{p_i} < K$$

So, we get $x \in w^\theta_\infty(p) \Delta$

ii) $\Rightarrow$ iii): Let (ii) holds. Suppose that (iii) does not holds. Then for some $t > 0$

$$\sup_r h^{r-1}_r \sum_{i \in I_r} M(t)^{p_i} = \infty$$
and therefore we can find a subinterval $I_{r(m)}$ of the set of interval $I_r$ such that
\begin{equation}
\tag{1.5}
h^{-1}_{r(m)} \sum_{i \in I_{r(m)}} M \left( \frac{1}{m} \right)^{p_i} > m, m = 1, 2, 3, \ldots
\end{equation}

Let us define $x = (x_i)$ as following
\[
\Delta x_i = \begin{cases} \frac{\rho}{m} & ; \ i \in I_{r(m)} \\ 0 & ; \ i \notin I_{r(m)} \end{cases}
\]

Then $x \in w^\rho_0 (p)_\Delta$ but by (5), $x \notin w^\rho_\infty (M, p)_\Delta$, which contradicts (ii).

Hence (iii) must holds.

**Proof.** (iii) $\Rightarrow$ i): Let (iii) hold and $x \in w^\rho_\infty (p)_\Delta$. Suppose that $x \notin w^\rho_\infty (M, p)_\Delta$.

Then for $x \in w^\rho_\infty (p)_\Delta$
\begin{equation}
\tag{1.6}
\sup_r h^{-1}_{r} \sum_{i \in I_r} M \left( \frac{\Delta x_i}{\rho} \right)^{p_i} = \infty
\end{equation}

Let $t = \frac{|\Delta x_i|}{\rho}$ for each $i$, then by (6)
\[
\sup_r h^{-1}_{r} \sum_{i \in I_r} M (t)^{p_i} = \infty
\]

which contradicts (iii). Hence (i) must holds.

**Theorem 2.6.** Let $M$ be an Orlicz function. Then the following statements are equivalent.

i) $w^\rho_0 (M, p)_\Delta \subset w^\rho_0 (p)_\Delta$

ii) $w^\rho_0 (M, p)_\Delta \subset w^\rho_\infty (p)_\Delta$

iii) $\inf_r h^{-1}_{r} \sum_{i \in I_r} M (t)^{p_i} > 0$ for all $t > 0$.

**Proof.** i) $\Rightarrow$ ii): It is obvious.

ii) $\Rightarrow$ iii): Let (ii) holds. Suppose that (iii) does not holds. Then
\[
\inf_r h^{-1}_{r} \sum_{i \in I_r} M (t)^{p_i} = 0
\]

and we can find a subinterval $I_{r(m)}$ of the set of interval $I_r$ such that
\begin{equation}
\tag{1.7}
h^{-1}_{r(m)} \sum_{i \in I_{r(m)}} M (m)^{p_i} < \frac{1}{m}, m = 1, 2, 3, \ldots
\end{equation}

Let us define $x = (x_i)$ as following
\[
\Delta x_i = \begin{cases} \rho m & ; \ i \in I_{r(m)} \\ 0 & ; \ i \notin I_{r(m)} \end{cases}
\]

Thus, by (7) $x \in w^\rho_0 (M, p)_\Delta$ but $x \notin w^\rho_\infty (p)_\Delta$ which contradicts (ii). Hence (iii) must holds.
iii) $\Rightarrow$ i): Let (iii) holds. Suppose that $x \in \mathcal{w}^0(M, p)$. Therefore,

$$\lim_{r \to \infty} r^{-1} \sum_{i \in I_r} M \left( \frac{\Delta x_i}{\rho} \right)^{p_i} = 0$$

as $r \to \infty$. Again, suppose that $x \notin \mathcal{w}^0(p)$ for some number $\varepsilon > 0$ and a subinterval $I_{r(m)}$ of the set of interval $I_r$, we have $\frac{\Delta x_i}{\rho} \geq \varepsilon$ for all $i$. Then, from properties of the Orlicz function, we can write

$$M \left( \frac{\Delta x_i}{\rho} \right)^{p_i} \geq M(\varepsilon)^{p_i}$$

Consequently, by (8) we have

$$\lim_{r \to \infty} r^{-1} \sum_{i \in I_r} M(\varepsilon)^{p_i} = 0$$

which contradicts (iii). Hence (i) must holds.

Finally, in this section, we consider that $(p_i)$ and $(q_i)$ are any bounded sequences of strictly positive real numbers. We are able to prove $\mathcal{w}^0(M, q) \subseteq \mathcal{w}^0(M, p)$ only under additional conditions.

**Theorem 2.7.**

i) If $0 < \inf p_i \leq p_i \leq 1$ for all $k$, then $\mathcal{w}^0(M) \subseteq \mathcal{w}^0(M, p)$

ii) $1 \leq p_i \leq \sup p_i = H < \infty$, then $\mathcal{w}^0(M, p) \subseteq \mathcal{w}^0(M)$

**Proof.** i) Let $x \in \mathcal{w}^0(M, p)$ since $0 < \inf p_i \leq p_i \leq 1$ we get

$$r^{-1} \sum_{i \in I_r} M \left( \frac{\Delta x_i - s_i}{\rho} \right) \leq r^{-1} \sum_{i \in I_r} M \left( \frac{\Delta x_i - s_i}{\rho} \right)^{p_i}$$

and hence $x \in \mathcal{w}^0(M)$. Let $1 \leq p_i \leq \sup p_i = H < \infty$, and $x \in \mathcal{w}^0(M)$. Then for each $0 < \varepsilon < 1$ there exists a positive integer $r_0$ such that

$$r^{-1} \sum_{i \in I_r} M \left( \frac{\Delta x_i - s_i}{\rho} \right) \leq \varepsilon < 1$$

for all $r \geq r_0$. This implies that

$$r^{-1} \sum_{i \in I_r} M \left( \frac{\Delta x_i - s_i}{\rho} \right)^{p_i} \leq r^{-1} \sum_{i \in I_r} M \left( \frac{\Delta x_i - s_i}{\rho} \right)^{p_i}$$

Therefore $x \in \mathcal{w}^0(M, p)$.

Using the same technique as in Theorem 2 in [14], it is easy to prove the following theorem.

**Theorem 2.8.** Let $0 < p_i \leq q_i$ for all $i$ and let $(q_i / p_i)$ be bounded. Then

$$\mathcal{w}^0(M, q) \subseteq \mathcal{w}^0(M, p)$$
REFERENCES


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