ON A GENERALIZATION OF NORMAL, ALMOST NORMAL AND MILDLY NORMAL SPACES II

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ABSTRACT. The aim of this paper is to study further characterizations and the relationships of $\delta p$-normal spaces, almost $\delta p$-normal spaces and mildly $\delta p$-normal spaces. We introduce the notion of $g\delta pr$-closed sets. Also, we obtain properties of $g\delta pr$-closed sets and the relationships between $g\delta pr$-closed sets and the related generalized closed sets. By using $g\delta pr$-closed sets, we introduce new forms of generalized $\delta$-precontinuity. Moreover, we obtain new characterizations of $\delta p$-normal spaces, almost $\delta p$-normal spaces and mildly $\delta p$-normal spaces and preservation theorems.

1. Introduction

In 2005, Ekici and Noiri [6] has introduced the notions of $\delta p$-normal spaces, almost $\delta p$-normal spaces and mildly $\delta p$-normal spaces. It is well known that separation axioms on topological spaces are important and basic subjects in studies of general topology and several branches of mathematics. In the literature, separation axioms have been researched by many mathematicians. Moreover, many authors have obtained many characterizations and generalizations of separation axioms by using generalized closed sets. In 1970, the first step of generalizing closed sets was done by Levine [9]. After that time, many authors have introduced and studied the relationships between separation axioms and generalized closed sets [13, 14]. The notions of generalized closed sets have been investigated extensively by many authors.
because the notion of generalized closed sets is a natural generalization of closed sets.

The purpose of this paper is to introduce a new class of generalized closed sets, namely \(g\delta pr\)-closed sets, which is a generalizing of \(g\delta p\)-closed sets and \(gpr\)-closed sets. The relations with other notions connected with \(g\delta pr\)-closed sets and also properties of \(g\delta pr\)-closed sets are investigated. Also, we introduce and study \(\delta p\)-regular \(T_{1/2}\) spaces and new forms of generalized \(\delta\)-precontinuous functions. As applications, using the notions of \(g\delta pr\)-closed sets, we obtain the further characterizations and properties of almost \(\delta p\)-normal spaces and mildly \(\delta p\)-normal spaces.

2. Preliminaries

Throughout this paper, spaces \((X, \tau)\) and \((Y, \sigma)\) (or simply \(X\) and \(Y\)) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let \(A\) be a subset of a space \(X\). The closure of \(A\) and the interior of \(A\) are denoted by \(cl(A)\) and \(int(A)\), respectively.

**Definition 1.** A subset \(A\) of a space \(X\) is said to be:

1. regular open [19] if \(A = int(cl(A))\),
2. \(\alpha\)-open [12] if \(A \subset int(cl(int(A)))\),

The complement of a preopen (resp. regular open) set is called preclosed [11] (resp. regular closed [19]). The intersection of all preclosed sets containing \(A\) is called the preclosure of \(A\) and is denoted by \(pcl(A)\). The preinterior of \(A\), denoted by \(pint(A)\) is defined to be the union of all preopen sets contained in \(A\).

The \(\delta\)-interior [20] of a subset \(A\) of \(X\) is defined by the union of all regular open sets of \(X\) contained in \(A\) and is denoted by \(\delta-int(A)\). A subset \(A\) is called \(\delta\)-open [20] if \(A = \delta-int(A)\), i.e. a set is \(\delta\)-open if it is the union of regular open sets. The complement of a \(\delta\)-open set is called \(\delta\)-closed. Alternatively, a set \(A\) of \((X, \tau)\) is called \(\delta\)-closed [20] if \(A = \delta-cl(A)\), where \(\delta-cl(A) = \{x \in X : A \cap int(cl(U)) \neq \emptyset, U \in \tau\text{ and }x \in U\}\).

A subset \(A\) of a space \(X\) is said to be \(\delta\)-preopen [17] if \(A \subset int(\delta-cl(A))\). The complement of a \(\delta\)-preopen set is said to be \(\delta\)-preclosed. The intersection of all \(\delta\)-preclosed sets of \(X\) containing \(A\) is called the \(\delta\)-preclosure [17] of \(A\) and is denoted by \(\delta-pcl(A)\). The union of all \(\delta\)-preopen sets of \(X\) contained in \(A\) is called \(\delta\)-preinterior of \(A\) and is denoted by \(\delta-pint(A)\) [17]. A subset \(U\) of \(X\) is called a \(\delta\)-preneighborhood of a point \(x \in X\) if there exists a \(\delta\)-preopen set \(V\) such that \(x \in V \subset U\). Note that \(\delta-pcl(A) = A \cup cl(\delta-int(A))\) and \(\delta-pint(A) = A \cap int(\delta-cl(A))\).

The family of all \(\delta\)-preopen (resp. \(\delta\)-preclosed, \(\alpha\)-open, \(\delta\)-open, \(\delta\)-closed) sets of a space \(X\) is denoted by \(\delta PO(X)\) (resp. \(\delta PC(X)\), \(\alpha O(X)\), \(\delta O(X)\), \(\delta C(X)\)).
Definition 2. A function \( f : X \rightarrow Y \) is called

(1) almost continuous \([18]\) (resp. R-map \([4]\), completely continuous \([1]\)) if \( f^{-1}(V) \) is open (resp. regular open, regular open) in \( X \) for every regular open (resp. regular open, open) set \( V \) of \( Y \),

(2) rc-preserving \([13]\) (resp. almost closed \([18]\)) if \( f(F) \) is regular closed (resp. closed) in \( Y \) for every regular closed set \( F \) of \( X \).

Definition 3. A subset \( A \) of a space \( X \) is called generalized closed \([9]\) (briefly, \( g \)-closed) if \( \text{cl}(A) \subset U \) whenever \( A \subset U \) and \( U \) is open in \( X \).

Definition 4. A subset \( A \) of a space \( X \) is called generalized p-closed \([10]\) (briefly, \( gp \)-closed) if \( p\text{cl}(A) \subset U \) whenever \( A \subset U \) and \( U \) is open in \( X \).

Definition 5. A subset \( A \) of a space \( X \) is called regular generalized closed \([16]\) (briefly, \( rg \)-closed) if \( \text{cl}(A) \subset U \) whenever \( A \subset U \) and \( U \) is regular open in \( X \).

Definition 6. A subset \( A \) of a space \( X \) is called generalized preregular closed \([8]\) or regular generalized preclosed \([14]\) (briefly, \( gpr \)-closed) if \( p\text{cl}(A) \subset U \) whenever \( A \subset U \) and \( U \) is regular open in \( X \).

The complement of a \( gp \)-closed (resp. \( rg \)-closed, \( gpr \)-closed) set is called \( gp \)-open (resp. \( rg \)-open, \( gpr \)-open).

3. \( g\delta p \)-closed sets

Definition 7. A subset \( A \) of a space \( X \) is called generalized \( \delta \)-preclosed (briefly, \( g\delta p \)-closed) \([6]\) if \( \delta \text{-pcl}(A) \subset U \) whenever \( A \subset U \) and \( U \) is open in \( X \).

Definition 8. A subset \( A \) of a space \( X \) is called generalized \( \delta p \)-regular closed (briefly, \( g\delta pr \)-closed) if \( \delta \text{-pcl}(A) \subset U \) whenever \( A \subset U \) and \( U \) is regular open in \( X \).

The complement of a \( g\delta p \)-closed (resp. \( g\delta pr \)-closed) set is called \( g\delta p \)-open (resp. \( g\delta pr \)-open).

The family of all generalized \( \delta p \)-regular open (resp. generalized \( \delta p \)-regular closed) sets of a space \( X \) is denoted by \( G\delta PRO(X) \) (resp. \( G\delta PREC(X) \)).

Remark 9. For a subset \( A \) of a topological space \((X, \tau)\), the following diagram holds:

\[
\begin{array}{ccc}
\text{closed} & \rightarrow & g\text{-closed} \\
\downarrow & & \downarrow \\
\text{preclosed} & \rightarrow & gp\text{-closed} \\
\downarrow & & \downarrow \\
\delta\text{-preclosed} & \rightarrow & g\delta p\text{-closed} \\
& & \downarrow \\
& & g\delta pr\text{-closed}
\end{array}
\]

None of these implications is reversible as shown by the following examples.
Example 10. Let \( X = \{a, b, c, d\} \) and \( \tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\} \). Then the sets \( \{a\} \) and \( \{b\} \) are \( \text{gδpr-closed} \) but not \( \text{gpr-closed} \).

Example 11. Let \( X = \{a, b, c, d\} \) and \( \tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\} \). Then the sets \( \{a, c\} \) and \( \{b, c\} \) are \( \text{gδpr-closed} \) but not \( \text{gδp-closed} \).

For the other implications the examples can be seen in [6, 8-11, 14, 16, 17].

Definition 12. A topological space \( X \) is said to be almost \( \text{δp-regular} \) if for each regular closed set \( A \) of \( X \) and each point \( x \in X \setminus A \), there exist disjoint \( \text{δ-preopen sets} U \) and \( V \) such that \( x \in U \) and \( A \subset V \).

Definition 13. A subset \( A \) of a topological space \( X \) is said to be \( \text{δ-preclosed relative to} \ X \) if for every cover \( \{V_\alpha : \alpha \in \Lambda\} \) of \( A \) by \( \text{δ-preopen subsets of} \ X \), there exists a finite subset \( \Lambda_0 \) of \( \Lambda \) such that \( A \subset \bigcup\{\text{δ-pcl}(V_\alpha) : \alpha \in \Lambda_0\} \).

Theorem 14. If a space \( X \) is almost \( \text{δp-regular} \) and a subset \( A \) of \( X \) is \( \text{δ-preclosed relative to} \ X \), then \( A \) is \( \text{gδpr-closed} \).

Proof. Let \( U \) be any regular open set of \( X \) containing \( A \). For each \( x \in A \), there exists a \( \text{δ-preopen set} V(x) \) such that \( x \in V(x) \subset \text{δ-pcl}(V(x)) \subset U \). Since \( \{V(x) : x \in A\} \) is a \( \text{δ-preopen cover of} \ A \), there exists a finite subset \( \Lambda_0 \) of \( \Lambda \) such that \( A \subset \bigcup\{\text{δ-pcl}(V(x)) : \alpha \in \Lambda_0\} \). Hence we obtain \( A \subset \text{δ- pcl}(A) \subset \bigcup\{\text{δ-pcl}(V(x)) : \alpha \in \Lambda_0\} \subset U \). This shows that \( A \) is \( \text{gδpr-closed} \). \( \square \)

Theorem 15. A subset \( A \) of a space \( X \) is \( \text{gδpr-open} \) if and only if \( M \subset \text{δ-pint}(A) \) whenever \( M \) is regular closed and \( M \subset A \).

Proof. \((\Rightarrow) : \) Let \( M \) be a regular closed set of \( X \) and \( M \subset A \). Then \( X \setminus M \) is regular open and \( X \setminus A \subset X \setminus M \). Since \( X \setminus A \) is \( \text{gδpr-closed} \), \( \text{δ-pcl}(X \setminus A) \subset X \setminus M \), i.e. \( X \setminus \text{δ-pint}(A) \subset X \setminus M \). Hence, \( M \subset \text{δ-pint}(A) \).

\((\Leftarrow) : \) Let \( G \) be a regular open set of \( X \) and \( X \setminus A \subset G \). Since \( X \setminus G \) is a regular closed set contained in \( A \), by hypothesis \( X \setminus G \subset \text{δ-pint}(A) \), i.e. \( X \setminus \text{δ-pint}(A) = \text{δ-pcl}(X \setminus A) \subset G \). Hence, \( X \setminus A \) is \( \text{gδpr-closed} \) and so \( A \) is \( \text{gδpr-open} \). \( \square \)

Theorem 16. A subset \( A \) of a space \( X \) is \( \text{gδp-open} \) if and only if \( F \subset \text{δ-pint}(A) \) whenever \( F \) is closed and \( F \subset A \).

Lemma 17. Let \( A \) and \( X_0 \) be subsets of a space \( (X, \tau) \). If \( A \in \text{δPO}(X) \) and \( X_0 \in \text{δO}(X) \), then \( A \cap X_0 \in \text{δPO}(X_0) \) [17].

Lemma 18. Let \( A \subset X \subset X_0 \). If \( X_0 \in \text{δO}(X) \) and \( A \in \text{δPO}(X_0) \), then \( A \in \text{δPO}(X) \) [17].
Theorem 19. If $Y$ is a regular open subspace of a space $X$ and $A$ is a subset of $Y$, then $\delta\text{-pcl}_Y(A) = \delta\text{-pcl}(A) \cap Y$.

Theorem 20. If $A$ is a regular open and $g\delta\text{pr}$-closed subset of a space $X$, then $A$ is $\delta\text{-preclosed in } X$.

Proof. If $A$ is regular open and $g\delta\text{pr}$-closed, then $\delta\text{-pcl}(A) \subset A$. This implies $A$ is $\delta\text{-preclosed}$.

Theorem 21. Let $Y$ be a regular open subspace of a space $X$ and $A \subset Y$. If $A$ is $g\delta\text{pr}$-closed in $X$, then $A$ is $g\delta\text{pr}$-closed in $Y$.

Proof. Let $U$ be a regular open set of $Y$ such that $A \subset U$. Then $U = V \cap Y$ for some regular open set $V$ of $X$. Since $A$ is $g\delta\text{pr}$-closed in $X$, we have $\delta\text{-pcl}(A) \subset V$ and by Theorem 19, $\delta\text{-pcl}_Y(A) = \delta\text{-pcl}(A) \cap Y \subset V \cap Y = U$. Hence, $A$ is $g\delta\text{pr}$-closed in $Y$.

Theorem 22. Let $Y$ be a $g\delta\text{pr}$-closed and regular open subspace of a space $X$. If $A$ is $g\delta\text{pr}$-closed in $Y$, then $A$ is $g\delta\text{pr}$-closed in $X$.

Proof. Let $U$ be any regular open subset of $X$ such that $A \subset U$. Since $U \cap Y$ is regular open in $Y$ and $A$ is $g\delta\text{pr}$-closed in $Y$, $\delta\text{-pcl}_Y(A) \subset U \cap Y$. By Theorem 19 and 20, we have $\delta\text{-pcl}(A) = \delta\text{-pcl}(A) \cap Y = \delta\text{-pcl}_Y(A) \subset U \cap Y \subset U$. Hence, $A$ is $g\delta\text{pr}$-closed in $X$.

Theorem 23. Let $Y$ be a regular open and $g\delta\text{pr}$-closed subspace of a space $X$. If $A$ is $g\delta\text{pr}$-open in $Y$, then $A$ is $g\delta\text{pr}$-open in $X$.

Proof. Let $M$ be any regular closed set and $M \subset A$. Since $M$ is regular closed in $Y$ and $A$ is $g\delta\text{pr}$-open in $Y$, $M \subset \delta\text{-pint}_Y(A)$ and then $M \subset \delta\text{-pint}(A) \cap Y$. Hence $M \subset \delta\text{-pint}(A)$ and so $A$ is $g\delta\text{pr}$-open in $X$.

Theorem 24. If a subset $A$ of a topological space $X$ is $g\delta\text{pr}$-closed, then $\delta\text{-pcl}(A) \setminus A$ contains no nonempty regular closed set in $X$.

Proof. Suppose that there exists a nonempty regular closed set $M$ of $X$ such that $M \subset \delta\text{-pcl}(A) \setminus A$. Since $X \setminus M$ is regular open and $A$ is $g\delta\text{pr}$-closed, $\delta\text{-pcl}(A) \subset X \setminus M$, i.e. $M \subset X \setminus \delta\text{-pcl}(A)$. Then $M \subset \delta - \text{pcl}(A) \cap (X \setminus \delta\text{-pcl}(A)) = \emptyset$ and hence $M = \emptyset$, which is a contradiction.

Theorem 25. A $g\delta\text{pr}$-closed subset $A$ of a topological space $X$ is $\delta\text{-preclosed}$ if and only if $\delta\text{-pcl}(A) \setminus A$ is regular closed.

Proof. Let $A$ be a $g\delta\text{pr}$-closed subset of $X$. Since $\delta\text{-pcl}(A) \setminus A$ is regular closed, by the previous theorem, $\delta\text{-pcl}(A) \setminus A = \emptyset$. Hence, $A$ is $\delta\text{-preclosed}$.

Conversely, if $g\delta\text{pr}$-closed set $A$ is $\delta\text{-preclosed}$, then $\delta\text{-pcl}(A) \setminus A = \emptyset$ and hence $\delta\text{-pcl}(A) \setminus A$ is a regular closed.
Theorem 26. If a subset $A$ of a topological space $X$ is $g\delta pr$-open, then $G = X$, whenever $G$ is regular open and $\delta - pint(A) \cup (X \setminus A) \subseteq G$.

Proof. Let $G$ be a regular open set of $X$ and $\delta - pint(A) \cup (X \setminus A) \subseteq G$. Then $X \setminus G \subseteq (X \setminus \delta - pint(A)) \cap A = \delta - pcl(X \setminus A) \setminus (X \setminus A)$. Since $X \setminus G$ is regular closed and $X \setminus A$ is $g\delta pr$-closed, by Theorem 24, $X \setminus G = \emptyset$ and hence $G = X$. □

Theorem 27. If a subset $A$ is $g\delta pr$-closed in a topological space $X$, then $\delta - pcl(A) \setminus A$ is $g\delta pr$-open.

Proof. Let $F$ be a regular closed set of $X$ such that $F \subseteq \delta - pcl(A) \setminus A$. Since $A$ is $g\delta pr$-closed, by Theorem 24, $F = \emptyset$ and hence $F \subseteq \delta - pint(\delta - pcl(A) \setminus A)$. Then by Theorem 15, $\delta - pcl(A) \setminus A$ is $g\delta pr$-open. □

4. $g\delta pr$-Closure of a Set

Theorem 28. Let $A$ and $B$ be $g\delta pr$-closed sets in $(X, \tau)$ such that $cl(A) = \delta - pcl(A)$ and $cl(B) = \delta - pcl(B)$. Then $A \cup B$ is $g\delta pr$-closed.

Proof. Let $A \cup B \subseteq U$ where $U$ is regular open. Then $A \subseteq U$ and $B \subseteq U$. Since $A$ and $B$ are $g\delta pr$-closed $\delta - pcl(A) \subseteq U$ and $\delta - pcl(B) \subseteq U$. Now, $cl(A \cup B) = cl(A) \cup cl(B) = \delta - pcl(A) \cup \delta - pcl(B) \subseteq U$. But $\delta - pcl(A \cup B) \subseteq cl(A \cup B)$. So $\delta - pcl(A \cup B) \subseteq U$ and hence $A \cup B$ is $g\delta pr$-closed. □

Theorem 29. If $A$ is $g\delta pr$-closed and $A \subseteq B \subseteq \delta - pcl(A)$, then $B$ is $g\delta pr$-closed.

Proof. Let $B \subseteq U$ where $U$ is regular open. Then $A \subseteq B$ implies $A \subseteq U$. Since $A$ is $g\delta pr$-closed, $\delta - pcl(A) \subseteq U$. $B \subseteq \delta - pcl(A)$ implies $\delta - pcl(B) \subseteq \delta - pcl(A)$. Thus, $\delta - pcl(B) \subseteq U$ and this shows that $B$ is $g\delta pr$-closed. □

Theorem 30. If $\delta - pint(A) \subseteq B \subseteq A$ and $A$ is $g\delta pr$-open, then $B$ is $g\delta pr$-open.

Proof. $\delta - pint(A) \subseteq B \subseteq A$ implies $X \setminus A \subseteq X \setminus B \subseteq X \setminus \delta - pint(A)$. That is, $X \setminus A \subseteq X \setminus B \subseteq \delta - pcl(X \setminus A)$. Since $X \setminus A$ is $g\delta pr$-closed, by Theorem 29, $X \setminus B$ is $g\delta pr$-closed and $B$ is $g\delta pr$-open. □

Definition 31. Let $(X, \tau)$ be a topological space and $A \subseteq X$. The almost kernel of $A$, denoted by $a - ker(A)$, is the intersection of all regular open supersets of $A$.

Theorem 32. A subset $A$ of a topological space $X$ is $g\delta pr$-closed if and only if $\delta - pcl(A) \subseteq a - ker(A)$.

Proof. Since $A$ is $g\delta pr$-closed, $\delta - pcl(A) \subseteq G$ for any regular open set $G$ with $A \subseteq G$ and hence $\delta - pcl(A) \subseteq a - ker(A)$.

Conversely, let $G$ be any regular open set such that $A \subseteq G$. By hypothesis, $\delta - pcl(A) \subseteq a - ker(A) \subseteq G$ and hence $A$ is $g\delta pr$-closed. □
Definition 33. For a subset $A$ of a topological space $(X, \tau)$, $g\delta pr\text{-}cl(A) = \cap\{F : A \subseteq F, F \text{ is } g\delta pr\text{-}closed \text{ in } X\}$.

Theorem 34. For a $x \in X$, $x \in g\delta pr\text{-}cl(A)$ if and only if $V \cap A \neq \emptyset$ for every $g\delta pr\text{-}open$ set $V$ containing $x$.

Proof. Suppose that there exists a $g\delta pr\text{-}open$ set $V$ containing $x$ such that $V \cap A = \emptyset$. Since $A \subseteq X \setminus V$, $g\delta pr\text{-}cl(A) \subseteq X \setminus V$. This implies $x \notin g\delta pr\text{-}cl(A)$, a contradiction.

Conversely, suppose that $x \notin g\delta pr\text{-}cl(A)$. Then there exists a $g\delta pr\text{-}closed$ subset $F$ containing $A$ such that $x \notin F$. Then $x \in X \setminus F$ and $X \setminus F$ is $g\delta pr\text{-}open$. Also $(X \setminus F) \cap A = \emptyset$, a contradiction. \qed

Theorem 35. Let $A$ and $B$ be subsets of a topological space $(X, \tau)$. Then:

1. $g\delta pr\text{-}cl(\emptyset) = \emptyset$ and $g\delta pr\text{-}cl(X) = X$,
2. If $A \subseteq B$, then $g\delta pr\text{-}cl(A) \subseteq g\delta pr\text{-}cl(B)$,
3. $A \subseteq g\delta pr\text{-}cl(A)$,
4. $g\delta pr\text{-}cl(A) = g\delta pr\text{-}cl(g\delta pr - cl(A))$,
5. $g\delta pr\text{-}cl(A \cup B) \supset g\delta pr\text{-}cl(A) \cup g\delta pr\text{-}cl(B)$,
6. $g\delta pr\text{-}cl(A \cap B) \subseteq g\delta pr\text{-}cl(A) \cap g\delta pr\text{-}cl(B)$.

Remark 36. If a subset $A$ of a space $X$ is $g\delta pr\text{-}closed$, then $g\delta pr\text{-}cl(A) = A$.

Theorem 37. Let $(X, \tau)$ be a topological space. If $G\delta PRO(X)$ is a topology, then $\tau^* = \{V \subseteq X : g\delta pr - cl(X \setminus V) = X \setminus V\} = G\delta PRO(X)$.

Theorem 38. For a topological space $(X, \tau)$, every $g\delta pr\text{-}closed$ set is closed if and only if $\tau^* = \tau$.

Proof. Let $A \in \tau^*$. Then $g\delta pr\text{-}cl(X \setminus A) = X \setminus A$. Since every $g\delta pr\text{-}closed$ set is closed, $cl(X \setminus A) = g\delta pr\text{-}cl(X \setminus A) = X \setminus A$. Hence, $A \in \tau$.

Conversely, suppose $\tau^* = \tau$. Let $A$ be a $g\delta pr\text{-}closed$ set. Then $g\delta pr\text{-}cl(A) = A$. This implies $X \setminus A \in \tau^* = \tau$. So $A$ is closed. \qed

5. $\delta p\text{-}regular \ T_{1/2}$ spaces and generalized $\delta\text{-}precontinuous$ functions

Definition 39. A topological space $X$ is called $\delta p\text{-}regular \ T_{1/2}$ if every $g\delta pr\text{-}closed$ set is $\delta\text{-}preclosed$.

Theorem 40. The following conditions are equivalent for a topological space $X$:

(a) $X$ is $\delta p\text{-}regular \ T_{1/2}$,
(b) Every singleton is either regular closed or $\delta\text{-}preopen$.

Proof. (a) $\Rightarrow$ (b) : Let $x \in X$ and assume that $\{x\}$ is not regular closed. Then $X \setminus \{x\}$ is not regular open and hence $X \setminus \{x\}$ is trivially $g\delta pr\text{-}closed$. By (a), it is $\delta\text{-}preclosed$ and hence $\{x\}$ is $\delta\text{-}preopen$. 

(b) ⇒ (a) : Let \( A \subseteq X \) be \( g\delta pr \)-closed. Let \( x \in \delta pcl(A) \). We will show that \( x \in A \). For, consider the following two cases:

Case (i)- The set \( \{x\} \) is regular closed. Then, if \( x \notin A \), there exists a regular closed set in \( \delta pcl(A) \setminus A \). By Theorem 24, \( x \in A \).

Case (ii)- The set \( \{x\} \) is \( \delta \)-preopen. Since \( x \in \delta pcl(A) \), then \( \{x\} \cap A \neq \emptyset \). Thus, \( x \in A \).

So, in both cases, \( x \in A \). This shows that \( \delta pcl(A) \subseteq A \) or equivalently \( A \) is \( \delta \)-preclosed. \( \square \)

**Theorem 41.** For a topological space \((X, \tau)\), the following hold:

1. \( \delta PO(X) \subseteq G\delta PRO(X) \),
2. The space \( X \) is \( \delta p \)-regular \( T_{1/2} \) if and only if \( \delta PO(X) = G\delta PRO(X) \).

**Proof.**

(1) Let \( A \) be \( \delta \)-preopen. Then \( X \setminus A \) is \( \delta \)-preclosed and so \( g\delta pr \)-closed. This implies that \( A \) is \( g\delta pr \)-open. Hence, \( \delta PO(X) \subseteq G\delta PRO(X) \).

(2) \( (\Rightarrow) \) : Let \( X \) be \( \delta p \)-regular \( T_{1/2} \). Let \( A \subseteq G\delta PRO(X) \). Then \( X \setminus A \) is \( g\delta pr \)-closed. By hypothesis, \( X \setminus A \) is \( \delta \)-preclosed and thus \( A \subseteq \delta PO(X) \).

Hence, \( G\delta PRO(X) = \delta PO(X) \).

(\( \Leftarrow \)) : Let \( \delta PO(X) = G\delta PRO(X) \). Let \( A \) be \( g\delta pr \)-closed. Then \( X \setminus A \) is \( g\delta pr \)-open. Hence, \( X \setminus A \subseteq \delta PO(X) \). Thus, \( A \) is \( \delta \)-preclosed thereby implying \( X \) is \( \delta \)-regular \( T_{1/2} \). \( \square \)

**Definition 42.** A function \( f : X \to Y \) is called:

1. \( g\delta p \)-continuous if \( f^{-1}(F) \) is \( g\delta p \)-closed in \( X \) for every \( \delta \)-preclosed set \( F \) of \( Y \),
2. \( \delta p \)-\( g\delta p \)-continuous \([6]\) if \( f^{-1}(F) \) is \( g\delta p \)-closed in \( X \) for every \( \delta \)-preclosed set \( F \) of \( Y \),
3. \( g\delta p \)-irresolute if \( f^{-1}(F) \) is \( g\delta p \)-closed in \( X \) for every \( \delta \)-preclosed set \( F \) of \( Y \).

**Definition 43.** A function \( f : X \to Y \) is called:

1. \( g\delta pr \)-continuous if \( f^{-1}(F) \) is \( g\delta pr \)-closed in \( X \) for every \( \delta \)-preclosed set \( F \) of \( Y \),
2. \( \delta p \)-\( g\delta pr \)-continuous if \( f^{-1}(F) \) is \( g\delta pr \)-closed in \( X \) for every \( \delta \)-preclosed set \( F \) of \( Y \),
3. \( g\delta pr \)-irresolute if \( f^{-1}(F) \) is \( g\delta pr \)-closed in \( X \) for every \( \delta \)-preclosed set \( F \) of \( Y \).

**Definition 44.** A function \( f : X \to Y \) is called:

1. \( \delta \)-precontinuous \([17]\) if \( f^{-1}(F) \) is \( \delta \)-preclosed in \( X \) for every \( \delta \)-preclosed set \( F \) of \( Y \),
2. \( \delta \)-preirresolute \([5]\) if \( f^{-1}(F) \) is \( \delta \)-preclosed in \( X \) for every \( \delta \)-preclosed set \( F \) of \( Y \).
Remark 45. The following diagram holds for a function \( f : (X, \tau) \to (Y, \sigma) \):

\[
\begin{array}{ccc}
g_\delta\text{-irresoluteness} & \Rightarrow & \delta p \Rightarrow \text{continuity} \\
\uparrow & & \uparrow \\
g_\delta \text{-irresoluteness} & \Rightarrow & \delta p \Rightarrow \text{continuity} \\
\uparrow & & \uparrow \\
\delta \text{-preirresoluteness} & \Rightarrow & \delta \text{-continuity}
\end{array}
\]

None of these implications is reversible as shown by the following examples.

Example 46. Let \( X = Y = \{a, b, c, d\} \) and \( \tau = \sigma = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\} \). Let \( f : (X, \tau) \to (Y, \sigma) \) be a function defined as follows: \( f(a) = a, f(b) = b, f(c) = b \) and \( f(d) = d \). Then \( f \) is \( g_\delta \) continuous but it is not \( g_\delta \) continuous.

If we define the function \( f : (X, \tau) \to (Y, \sigma) \) as follows: \( f(a) = b, f(b) = d, f(c) = a \) and \( f(d) = d \), then \( f \) is \( g_\delta \) continuous but it is not \( \delta p \)-\( g_\delta \) continuous. Also, \( f \) is \( \delta \)-precontinuous but it is not \( \delta \)-preirresolute.

Example 47. Let \( X = Y = \{a, b, c, d\} \) and \( \tau = \sigma = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\} \). Let \( f : (X, \tau) \to (Y, \sigma) \) be a function defined as follows: \( f(a) = d, f(b) = a, f(c) = b \) and \( f(d) = d \). Then \( f \) is \( \delta p \)-\( g_\delta \) continuous but it is not \( g_\delta \)-irresolute.

If we define the function \( f : (X, \tau) \to (Y, \sigma) \) as follows: \( f(a) = c, f(b) = c, f(c) = a \) and \( f(d) = a \), then \( f \) is \( \delta p \)-\( g_\delta \) continuous but it is not \( \delta p \)-\( g_\delta \) continuous.

Example 48. Let \( X = Y = \{a, b, c, d\} \) and \( \tau = \sigma = \{X, \emptyset, \{b\}, \{d\}, \{b, d\}\} \). Let \( f : (X, \tau) \to (Y, \sigma) \) be a function defined as follows: \( f(a) = a, f(b) = c, f(c) = b \) and \( f(d) = d \). Then \( f \) is \( g_\delta \)-continuous but it is neither \( \delta p \)-\( g_\delta \)-continuous nor \( \delta \)-precontinuous.

If we define the function \( f : (X, \tau) \to (Y, \sigma) \) as follows: \( f(a) = a, f(b) = b, f(c) = d \) and \( f(d) = b \), then \( f \) is \( \delta p \)-\( g_\delta \) continuous but it is neither \( g_\delta \)-irresolute nor \( \delta \)-preirresolute.

Theorem 49. Let \( f : X \to Y \) and \( g : Y \to Z \) be functions.

1. If \( f \) is \( g_\delta \)-irresolute and \( g \) is \( g_\delta \)-continuous, then the composition \( g \circ f : X \to Z \) is \( g_\delta \)-continuous.
2. If \( f \) is \( g_\delta \)-irresolute and \( g \) is \( \delta p \)-\( g_\delta \) continuous, then the composition \( g \circ f : X \to Z \) is \( \delta p \)-\( g_\delta \)-continuous.
3. If \( f \) and \( g \) are \( \delta p \)-\( g_\delta \)-continuous and \( Y \) is \( \delta p \)-regular \( T_{1/2} \), then the composition \( g \circ f : X \to Z \) is \( \delta p \)-\( g_\delta \)-continuous.
4. If \( f \) and \( g \) are \( g_\delta \)-irresolute, then the composition \( g \circ f : X \to Z \) is \( g_\delta \)-irresolute.

Theorem 50. If a function \( f : X \to Y \) is \( \delta p \)-\( g_\delta \)-continuous and \( Y \) is \( \delta p \)-regular \( T_{1/2} \), then \( f \) is \( g_\delta \)-irresolute.
Proof. Let $F$ be any $gδpr$-closed subset of $Y$. Since $Y$ is $δp$-regular $T_{1/2}$, then $F$ is $δ$-preclosed in $Y$. Hence $f^{-1}(F)$ is $rgδp$-closed in $X$. This shows that $f$ is $gδpr$-irresolute. □

**Definition 51.** A topological space $X$ is said to be
(1) extremally disconnected [3] if the closure of each open set of $X$ is open in $X$,
(2) submaximal [3] if every dense subset of $X$ is open.

**Definition 52.** Let $(X, τ)$ be a topological space. The collection of all regular open sets forms a base for a topology $τ^*$. It is called the semiregularization. In case $τ = τ^*$, the space $(X, τ)$ is called semi-regular [19].

**Theorem 53.** If a function $f : X → Y$ is $gδpr$-continuous and $Y$ is submaximal extremally disconnected and semi-regular, then $f$ is $δp$-$gδpr$-continuous.

Proof. Let $F$ be any $δ$-preclosed subset of $Y$. Since $Y$ is submaximal extremally disconnected semi-regular, then $F$ is closed in $Y$. Hence $f^{-1}(F)$ is $rgδp$-closed in $X$. This shows that $f$ is $δp$-$gδpr$-continuous. □

**Theorem 54.** If a function $f : X → Y$ is $gδpr$-continuous and $X$ is $δp$-regular $T_{1/2}$, then $f$ is $δ$-precontinuous.

Proof. Let $F$ be any closed set of $Y$. Since $f$ is $gδpr$-continuous, $f^{-1}(F)$ is $gδpr$-closed in $X$ and then $f^{-1}(F)$ is $δ$-preclosed in $X$. Hence $f$ is $δ$-precontinuous. □

**Theorem 55.** If a function $f : X → Y$ is $δp$-$gδpr$-continuous and $X$ is $δp$-regular $T_{1/2}$, then $f$ is $δ$-preirresolute.

Proof. Let $F$ be any $δ$-preclosed set of $Y$. Since $f$ is $δp$-$gδpr$-continuous, $f^{-1}(F)$ is $gδpr$-closed in $X$ and then $f^{-1}(F)$ is $δ$-preclosed in $X$. Hence $f$ is $δ$-preirresolute. □

6. Almost $δp$-normal, mildly $δp$-normal spaces and preservation properties

**Definition 56.** A function $f : X → Y$ is called strongly $δ$-preclosed [6] if $f(U) ∈ δPC(Y)$ for each $U ∈ δPC(X)$.

**Definition 57.** A function $f : X → Y$ is called
(1) $δp$-$gδp$-closed [6] if $f(F)$ is $gδp$-closed in $Y$ for every $δ$-preclosed set $F$ of $X$,
(2) $δp$-$gδpr$-closed if $f(F)$ is $gδpr$-closed in $Y$ for every $δ$-preclosed set $F$ of $X$. 
Definition 58. A space $X$ is said to be

(1) δp-normal [6] if for every pair of disjoint closed sets $A$ and $B$ of $X$, there exist disjoint δ-preopen sets $U$ and $V$ such that $A \subset U$ and $B \subset V$,

(2) almost δp-normal [6] if for each closed set $A$ and regular closed set $B$ of $X$ such that $A \cap B = \emptyset$, there exist disjoint δ-preopen sets $U$ and $V$ such that $A \subset U$ and $B \subset V$,

(3) mildly δp-normal [6] if for every pair of disjoint regular closed sets $A$ and $B$ of $X$, there exist disjoint δ-preopen sets $U$ and $V$ such that $A \subset U$ and $B \subset V$.

Theorem 59. The following are equivalent for a space $X$:

(1) $X$ is almost δp-normal,

(2) For each closed set $A$ and regular closed set $B$ such that $A \cap B = \emptyset$, there exist disjoint gδp-open sets $U$ and $V$ such that $A \subset U$ and $B \subset V$,

(3) For each closed set $A$ and each regular open set $B$ containing $A$, there exists a gδp-open set $V$ of $X$ such that $A \subset V \subset \delta\text{-pcl}(V) \subset B$,

(4) For each rg-closed set $A$ and each regular open set $B$ containing $A$, there exists a δ-preopen set $V$ of $X$ such that cl$(A) \subset V \subset \delta\text{-pcl}(V) \subset B$,

(5) For each rg-closed set $A$ and each regular open set $B$ containing $A$, there exists a gδp-open set $V$ of $X$ such that cl$(A) \subset V \subset \delta\text{-pcl}(V) \subset B$,

(6) For each $g$-closed set $A$ and each regular open set $B$ containing $A$, there exists a δ-preopen set $V$ of $X$ such that cl$(A) \subset V \subset \delta\text{-pcl}(V) \subset B$,

(7) For each $g$-closed set $A$ and each regular open set $B$ containing $A$, there exists a gδp-open set $V$ of $X$ such that cl$(A) \subset V \subset \delta\text{-pcl}(V) \subset B$.

Proof. (1) ⇒ (2), (3) ⇒ (5), (5) ⇒ (4), (4) ⇒ (6) and (6) ⇒ (7) are obvious.

(2) ⇒ (3) : Let $A$ be a closed set and $B$ be a regular open subset of $X$ containing $A$. There exist disjoint gδp-open sets $V$ and $W$ such that $A \subset V$ and $X \setminus B \subset W$. By Theorem 16, we have $X \setminus B \subset \delta\text{-pint}(W)$ and $V \cap \delta\text{-pint}(W) = \emptyset$. Hence, we obtain $\delta\text{-pcl}(V) \cap \delta\text{-pint}(W) = \emptyset$ and hence $A \subset V \subset \delta\text{-pcl}(V) \subset X \setminus \delta\text{-pint}(W) \subset B$.

(7) ⇒ (1) : Let $A$ be any closed set and $B$ be any regular closed set such that $A \cap B = \emptyset$. Then $X \setminus B$ is a regular open set containing $A$ and there exists a gδp-open set $G$ of $X$ such that $A \subset G \subset \delta\text{-pcl}(G) \subset X \setminus B$. Put $U = \delta\text{-pint}(G)$ and $V = X \setminus \delta\text{-pcl}(G)$. Then $U$ and $V$ are disjoint δ-preopen sets of $X$ such that $A \subset U$ and $B \subset V$. Hence $X$ is almost δp-normal.

Theorem 60. The following are equivalent for a space $X$:

(1) $X$ is mildly δp-normal,

(2) For any disjoint regular closed sets $A$ and $B$ of $X$, there exist disjoint gδp-open sets $U$ and $V$ such that $A \subset U$ and $B \subset V$,

(3) For any disjoint regular closed sets $A$ and $B$ of $X$, there exist disjoint gδp-open sets $U$ and $V$ such that $A \subset U$ and $B \subset V$,

(4) For each regular closed set $A$ and each regular open set $B$ containing $A$, there exists a gδp-open set $V$ of $X$ such that $A \subset V \subset \delta\text{-pcl}(V) \subset B$,
(5) For each regular closed set \( A \) and each regular open set \( B \) containing \( A \), there exists a \( g \delta \rho \)-open set \( V \) of \( X \) such that \( A \subset V \subset \delta -\text{pcl}(V) \subset B \).

**Proof.** The proof is similar to Theorem 59 by using Theorem 29 in [6]. □

**Theorem 61.** Let \( f : (X, \tau) \to (Y, \sigma) \) be \( R \)-map and strongly \( \delta \)-preclosed. Then for every \( g \delta \rho \)-closed set \( A \) of \( X \), \( f(A) \) is \( g \delta \rho \)-closed in \( Y \).

**Proof.** Let \( A \) be \( g \delta \rho \)-closed in \( X \). Let \( f(A) \subset U \), where \( U \) is regular open in \( Y \). Then \( A \subset f^{-1}(U) \). Since \( f \) is \( R \)-map and \( A \) is \( g \delta \rho \)-closed, \( \delta \)-\text{pcl}(\( A \)) \subset \( f^{-1}(U) \). That is, \( f(\delta \text{-pcl}(A)) \subset U \). Now \( \delta \text{-pcl}(f(A)) \subset \delta -\text{pcl}(f(\delta \text{-pcl}(A))) = f(\delta -\text{pcl}(A)) \subset U \), since \( f \) is strongly \( \delta \)-preclosed. Hence, \( f(A) \) is \( g \delta \rho \)-closed in \( Y \). □

**Lemma 62.** A surjection \( f : X \to Y \) is \( \delta \rho \)-\( g \delta \rho \)-closed (resp. \( \delta \rho \)-\( g \delta \rho \)-closed) if and only if for each subset \( B \) of \( Y \) and each \( \delta \)-preopen set \( U \) of \( X \) containing \( f^{-1}(B) \) there exists a \( \delta \rho \)-open (resp. \( g \delta \rho \)-open) set of \( V \) of \( Y \) such that \( B \subset V \) and \( f^{-1}(V) \subset U \).

**Theorem 63.** If \( f : X \to Y \) is a \( \delta \rho \)-\( g \delta \rho \)-closed continuous surjection and \( X \) is \( \delta \rho \)-normal, then \( Y \) is \( \delta \rho \)-normal.

**Proof.** Let \( A \) and \( B \) be any disjoint closed sets of \( Y \). Then \( f^{-1}(A) \) and \( f^{-1}(B) \) are disjoint closed sets of \( X \). Since \( X \) is \( \delta \rho \)-normal, there exist disjoint \( \delta \)-preopen sets \( U \) and \( V \) such that \( f^{-1}(A) \subset U \) and \( f^{-1}(B) \subset V \). By Lemma 62, there exist \( \delta \rho \)-open sets \( G \) and \( H \) of \( Y \) such that \( A \subset G \), \( B \subset H \), \( f^{-1}(G) \subset U \) and \( f^{-1}(H) \subset V \). Since \( U \) and \( V \) are disjoint, \( G \) and \( H \) are disjoint. By Theorem 16, we have \( A \subset \delta \text{-pint}(G) \), \( B \subset \delta \text{-pint}(H) \) and \( \delta \text{-pint}(G) \cap \delta \text{-pint}(H) = \varnothing \). This shows that \( Y \) is \( \delta \rho \)-normal. □

The proofs of the following theorems are similar to previous one.

**Theorem 64.** If \( f : X \to Y \) is an \( R \)-map \( \delta \rho \)-\( rg \delta \rho \)-closed surjection and \( X \) is mildly \( \delta \rho \)-normal, then \( Y \) is mildly \( \delta \rho \)-normal.

**Theorem 65.** If \( f : X \to Y \) is a completely continuous \( \delta \rho \)-\( rg \delta \rho \)-closed surjection and \( X \) is mildly \( \delta \rho \)-normal, then \( Y \) is \( \delta \rho \)-normal.

**Theorem 66.** If \( f : X \to Y \) is an almost continuous \( \delta \rho \)-\( rg \delta \rho \)-closed surjection and \( X \) is \( \delta \rho \)-normal, then \( Y \) is mildly \( \delta \rho \)-normal.

**Theorem 67.** If \( f \) is a \( \delta \rho \)-\( g \delta \rho \)-continuous \( rc \)-preserving injection and \( Y \) is mildly \( \delta \rho \)-normal, then \( X \) is mildly \( \delta \rho \)-normal.

**Proof.** Let \( A \) and \( B \) be any disjoint regular closed sets of \( X \). Since \( f \) is an \( rc \)-preserving injection, \( f(A) \) and \( f(B) \) are disjoint regular closed sets of \( Y \). By mild \( \delta \rho \)-normality of \( Y \), there exist disjoint \( \delta \)-preopen sets \( U \) and \( V \) of \( Y \) such that \( f(A) \subset U \) and \( f(B) \subset V \). Since \( f \) is \( \delta \rho \)-\( g \delta \rho \)-continuous, \( f^{-1}(U) \) and \( f^{-1}(V) \) are disjoint \( g \delta \rho \)-open sets containing \( A \) and \( B \), respectively. Hence by Theorem 60, \( X \) is mildly \( \delta \rho \)-normal. □
Theorem 68. If \( f : X \rightarrow Y \) is a \( \delta p-\delta pr \)-continuous almost closed injection and \( Y \) is \( \delta p \)-normal, then \( X \) is mildly \( \delta p \)-normal.

Proof. Similar to previous one. \( \square \)

References


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