Abstract: We sharpen and extend inequalities concerning generalized inverses previously obtained for the von Neumann-Schatten, and supremum, norms. We sharpen those inequalities to obtain corresponding inequalities for singular values \( s_i(\cdot) \) for \( i = 1, 2, \ldots \); and we extend those inequalities, for finite rank operators, to inequalities for an arbitrary unitarily invariant norm.

Received: December 24, 2006

Keywords and phrases: Generalized inverses, Moore-Penrose inverse, von Neumann-Schatten classes and norms.

2000 Mathematics Subject Classification: 47A05, 47B10.

1. INTRODUCTION

This paper sharpens and extends inequalities concerning generalized inverses previously obtained for the von Neumann-Schatten, and supremum, norms. It sharpens those inequalities to obtain corresponding inequalities for singular values and it extends them, at least for finite rank operators, to inequalities concerning an arbitrary unitarily invariant norm. For example, as is well-known [5, Theorems 2.1, 2.2], [6,
Theorem 3.3], if \( A \) has a (i), (iii) inverse \( A^- \) (meaning that if \( A \) satisfies \( AA^- = A \) and \( (AA^-)^\dagger = AA^- \)) and if \( AX - C \in C_p, \) where \( 1 \leq p < \infty \), then
\[
\| AX - C \|_p \geq \| AA^- C - C \|_p
\]
(1.1)

(where \( C_p \) denotes the von Neumann-Schatten class and \( \| \cdot \|_p \) its norm). Here, in Corollary 3.1 below, we prove that
\[
s_i(AX - C) \geq s_i(AA^- C - C) \quad \text{for } i = 1, 2, \ldots
\]
(1.2)

where the singular values \( s_i(AX - C) \) and \( s_i(AA^- C - C) \) are – as always – arranged in decreasing order and repeated according to multiplicity; and we prove that if \( AX - C \) is of finite rank then so is \( AA^- C - C \) and for every unitarily invariant norm \( \| \| \cdot \| \)
\[
\| AX - C \| \geq \| AA^- C - C \| .
\]
(1.3)

The finite rank condition is required – as far as unitarily invariant norms are concerned – so that we can appeal to the von Neumann-Schatten theory of unitarily invariant norms and their representation by symmetric gauge functions [8, Chapter 10], [12, Chapter V].

The singular values inequality (1.2) is deduced as a corollary to the main result of this paper, Theorem 3.1: this shows that a modulus inequality between operators implies one between singular values which, for finite rank operators, implies an inequality between unitarily invariant norms (just as in [6, Lemma 3.1] an inequality between moduli of operators implies one between von Neumann-Schatten norms). Another consequence of Theorem 3.1 is Theorem 3.2: this says that if \( A = PAQ = PXQ \), where \( A, P, Q \) are fixed and \( P \) and \( Q \) are projections, and if \( X \in C_p \) for \( 1 \leq p < \infty \) then, amongst other things, \( s_i(A) \geq s_i(X) \) for \( i = 1, 2, \ldots \).

Theorem 3.2 yields as corollaries (Corollaries 3.2, 3.3 and 3.4) a host of sharpenings/extensions of already known results to do with generalized inverses including the result, Corollary 3.4 [5, Theorems 3.1 and 3.2] that \( s_i(A^-) \geq s_i(A^+) \) for \( i = 1, 2, \ldots \), where \( A^- \) is a generalized inverse of \( A \) and \( A^+ \) is the Moore-Penrose inverse of \( A \).

Theorem 3.2 and Corollaries 3.2, 3.3 and 3.4 are, as pointed out below, essentially finite dimensional since their hypotheses imply that the operator \( A \) occurring in them (and its adjoint \( A^- \)) must be of finite rank.

For unitarily invariant norms (not necessarily on operators of finite rank) there is a property of strict convexity, analogous to the strict convexity of \( \| \cdot \|_p \) for \( 1 < p < \infty \); and, as with \( \| \cdot \|_p \), strict convexity implies a uniqueness property (stated in Lemma 2.3). Thus, in (3.3), for example, equality holds if, and for strictly convex \( \| \| \cdot \| \) only if, \( AX = AA^- C \).
2. PRELIMINARIES

Throughout, \( H \) is a complex, separable Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \); and all operators are in \( \mathcal{L}(H) \), the space of all bounded, linear operators mapping \( H \) to \( H \). An operator \( A \) is self-adjoint if \( A = A^* \); equivalently, if \( \langle Af, f \rangle \geq 0 \) for all \( f \in H \); a (self-adjoint) operator is positive, denoted \( A \geq 0 \), if \( A - B \geq 0 \). It can be shown that every positive operator \( T \) has a unique positive square root, denoted \( T^{\frac{1}{2}} \). The modulus \( |A| \) of an arbitrary operator \( A \) is the positive square root of (the positive operator) \( A^*A \); that is, \( |A| = (A^*A)^{\frac{1}{2}} \).

Recall that an operator \( A^- \) is a generalized inverse of \( A \) if \( AA^-A = A \). An operator \( A \) has a generalized inverse if and only if its range, Ran \( A \), is closed (We define Ran \( A = \{ Af : f \in H \} \) \[13, p.251, Theorem 12.9\]). For an operator \( A \), with closed range, its Moore-Penrose inverse \( A^+ \) satisfies

\[
\begin{align*}
AA^+A &= A \quad (i) \\
A^+AA^+ &= A^+ \quad (ii) \\
(AA^+)^* &= AA^+ \quad (iii) \\
(A^+A)^* &= A^+A \quad (iv);
\end{align*}
\]

and, further, \( A^- \) is uniquely determined by these properties \[9, Theorem 1\]. For the construction of the Moore-Penrose inverse of an operator with closed range see \[13, p.251, Theorem 12.9\]. If an operator \( A^- \) satisfies (i) and (iii) of (2.1) (so that \( AA^-A = A \) and \( (AA^-)^* = AA^- \)) it will be called a (i), (iii) inverse of \( A \); if \( A^- \) satisfies (i), (iv) of (2.1) it will be called a (i), (iv) inverse of \( A \). Observe that if \( A^- \) is a (i), (iii) inverse of \( A \) then \( AA^- \) is the projection onto Ran \( A \) and that if \( A^- \) is a (i), (iv) inverse of \( A \) then \( A^-A \) is the projection onto \( (\ker A)^{\perp} \).

An operator \( A \) is of finite rank \( n \), denoted rank \( A = n \), if \( \dim \text{Ran} \ A = n < \infty \). The rank 1 operator \( x \mapsto \langle x, f \rangle g \), for fixed vectors \( f \) and \( g \) in \( H \), is denoted \( f \otimes g \). The spectral theorem for compact, positive operators says that every compact, positive operator \( X \) can be expressed uniquely as

\[
X = \sum_i \alpha_i (f_i \otimes f_i) \quad (2.2)
\]
where \((\alpha_i)\) is the sequence of positive eigenvalues of \(X\) arranged in decreasing order and repeated according to multiplicity, and \((f_i)\) is the corresponding orthonormal sequence of eigenvectors (so that \(S\{f_i\} = \text{Ran} \ X\); and \(X\) is of finite rank \(n\) if and only if the sequence \((\alpha_i)\) terminates after just \(n\) terms [11, p.64 (2), p.70 cf. Theorem 1.9.3]. In particular, if \(X = \| A \|\), for some \(A\) in \(L(H)\), the eigenvalues \(\alpha_i\) are called the singular values of \(A\) and denoted \(s_i(A)\).

The spectral theorem for compact operators says that every compact operator \(X\) can be expressed uniquely as

\[
X = \sum_i s_i(X)(f_i \otimes g_i)
\]  

(2.3)

where \((f_i)\) and \((g_i)\) are orthonormal sequences in \(H\) and \((s_i(X))\) is the sequence of singular values of \(X\), arranged in decreasing order and repeated according to multiplicity; and \(X\) is of finite rank \(n\) if and only if the sequence \((s_i(X))\) terminates after just \(n\) terms [11, Theorem 1.9.3].

For a compact operator \(A\), let \(s_1(A), s_2(A), \ldots\) denote the singular values of \(A\), arranged in decreasing order and repeated according to multiplicity. If, for some \(p > 0\), \(\sum_{i=1}^{\infty} s_i^p(A) < \infty\) we say \(A\) is in the von Neumann-Schatten class \(C_p\) and write

\[
\| A \|_p = \left[ \sum_{i=1}^{\infty} s_i^p(A) \right]^{\frac{1}{p}}.
\]

For \(1 \leq p < \infty\), it can be shown that \(\| \cdot \|_p\) is a norm called the von Neumann-Schatten norm. We sometimes write \(\| \cdot \|_\infty = \| \cdot \|\), the supremum norm. For all \(p\), where \(0 < p < \infty\), \(C_p\) is a 2-sided ideal of \(L(H)\) and for \(1 \leq p < \infty\) the space \(C_p\) is a Banach space under \(\| \cdot \|_p\). For more details of the von Neumann-Schatten classes and norms see [2, Chapter XI], [11, Chapter 2].

The class \(C_1\) is called the trace class. If \(A \in C_1\) and \((\phi_i)\) is an orthonormal basis of \(H\) then the quantity \(\text{tr} \ A\), called the trace of \(A\) and defined by \(\text{tr} \ A = \sum_i \langle A\phi_i, \phi_i \rangle\), can be shown to be finite and independent of the particular basis \((\phi_i)\) chosen. Further, \(A \in C_p\), where \(1 \leq p < \infty\), if and only if \(\| A \| \in C_1\).

From (2.3) it follows that for every \(X\) in \(C_p\), where \(1 \leq p < \infty\), there exists a sequence \((X_n)\) of operators, each of finite rank \(n\), such that

\[
\| X - X_n \|_p \to 0 \quad \text{as} \quad n \to \infty.
\]

(2.4)
The von Neumann-Schatten norms \( \| \cdot \|_p \), for \( 1 \leq p < \infty \), satisfy the property of uniformity [2, Chap XI, 9.9(d)]: if \( A \in \mathcal{L}(H) \), \( B \in \mathcal{L}(H) \) and \( X \in \mathcal{C}_p \) then
\[
\| AXB \|_p \leq \| A \|_p \| X \|_p \| B \|_p .
\] (2.5)

We cite three further results we shall need about the von Neumann-Schatten norms.

**Lemma 2.1** [6, Lemma 3.1].
(a) If \( |A|^2 \geq |B|^2 \) then \( \|A\| \geq \|B\| \);
(b) if, further, \( A \) is compact then \( |B| \) is compact and \( |A| \geq |B| \);
(c) if, further, \( A \in \mathcal{C}_p \), for \( 1 \leq p < \infty \), then \( B \in \mathcal{C}_p \) and
\[
\|A\|_p \geq \|B\|_p .
\]

Recall that the polar decomposition [3, Chapter 16] says that every operator \( A \) can be expressed uniquely as \( A = U |A| \), where the partial isometry \( U \) is such that \( \text{Ker} U = \text{Ker} |A| \). (A partial isometry \( U \) satisfies \( \|Uf\| = \|f\| \) for all \( f \) in \( \text{Ker} U \).)

**Theorem 2.1** [1, Theorem 2.1]. If \( 1 < p < \infty \), the map \( X \mapsto \|X\|_p^p \) (from \( \mathcal{C}_p \) to \( \mathbb{R}^+ \)) is differentiable with derivative \( D_X(S) = p \text{Re} \text{tr} [X |X|^{p-1} U^* S] \) where \( X = U |X| \) is the polar decomposition of \( X \). If the underlying space \( H \) is finite-dimensional the same result holds for \( 0 < p < \infty \) at every invertible element \( X \).

**Lemma 2.2** [6, Lemma 2.5]. If \( S \) is a convex set of operators in \( \mathcal{C}_p \), where \( 1 < p < \infty \), there is at most one minimizer of \( \|X\|_p^p \) where \( X \in S \).

A norm \( \| \cdot \| \) is said to be unitarily invariant if \( \|AU\| = \|VA\| \) for all unitary operators \( U \) and \( V \) (provided \( \|A\| < \infty \)). Examples of unitarily invariant norms are the supremum norm \( \| \cdot \| \) and the von Neumann-Schatten norms \( \| \cdot \|_p \), \( 1 \leq p < \infty \).

A unitarily invariant norm \( \| \cdot \| \) is strictly convex if
\[
\|A + B\| = \|A\| + \|B\| \Rightarrow A = kB
\]
for real, positive \( k \) (provided \( \|A\| < \infty \) and \( \|B\| < \infty \)). The next result – a generalization of Lemma 2.2 – is proved similarly to it.
**LEMMA 2.3.** Let $||| \cdot |||$ be a strictly convex, unitarily invariant norm and let 
\[
\{X \in L(H) : ||| X ||| < \infty \} = \mathcal{S},
\]
say. Then if $S$ is a convex set of operators in $\mathcal{S}$ there is at most one minimizer of $||| X |||$ where $X \in S$.

Many properties of unitarily invariant norms can be deduced via Theorem 2.2 from those of symmetric gauge functions provided the operators concerned are of finite rank. References below are to Schatten’s own elegant exposition [12].

Let $F$ be the set of all operators of finite rank and let $L$ be the set of all sequences of real numbers having a finite number of non-zero terms. A *symmetric gauge function* $\phi : L \to R^+$ is a function satisfying the following properties:

\[
\begin{align*}
(i) & \quad \phi(u) > 0 \quad \text{if } u \neq 0 \\
(ii) & \quad \phi(\alpha u) = |\alpha| \phi(u) \\
(iii) & \quad \phi(u + v) \leq \phi(u) + \phi(v) \\
(iv) & \quad \phi(u_1, \ldots, u_n) = \phi(e_i u_i, \ldots, e_i u_i)
\end{align*}
\]

where each $e_i = \pm 1$ and $\{i_1, \ldots, i_n\}$ is a permutation of $\{1, \ldots, n\}$. (In (2.6) (iv), and below, we write $\phi(u_1, u_2, \ldots)$ instead of $\phi(u)$ for the values of the symmetric gauge function $\phi$.)

**LEMMA 2.4** [12, p.68, cf. p.61, Lemma 6]. Let $\phi$ be a symmetric gauge function. If $u_i \geq v_i \geq 0$ for $1 \leq i \leq n$ then

\[
\phi(u_1, \ldots, u_n, 0, \ldots) \geq \phi(v_1, \ldots, v_n, 0, \ldots).
\]

**THEOREM 2.2** [12, p.69, Theorem 8]. Let $A$ be in $F$ and let $s_1(A), s_2(A), \ldots$ be the singular values of $A$ arranged in decreasing order and repeated according to multiplicity. If $\phi : L \to R^+$ is a symmetric gauge function then the function

\[
\Phi : F \mapsto R^+
\]

\[
\Phi(A) = \phi(s_1(A), s_2(A), \ldots)
\]

is a unitarily invariant norm $||| \cdot ||| : F \mapsto \mathbb{R}^+$, i.e., $||| A ||| = \Phi(A)$; and, conversely, if $||| \cdot ||| : F \mapsto R^+$ is a unitarily invariant norm then there exists a symmetric gauge function $\phi$ such that

\[
||| A ||| = \phi(s_1(A), s_2(A), \ldots).
\]
It follows from Theorem 2.2, first, that every unitarily invariant norm \( \| \cdot \| \) is *self-adjoint*, i.e., \( \| A \| = \| A^* \| \) (because the non-zero eigenvalues of \( (A^*A)^{1/2} \) and \( (AA^*)^{1/2} \) are equal); and, second, that every unitarily invariant norm \( \| \cdot \| \) has the property of *uniformity*: if \( A \in L(H) \), \( B \in L(H) \) and \( X \in F \) then [12, p.71, Theorem 11]

\[
\| AXB \| \leq \| A \| \| X \| \| B \|. \tag{2.7}
\]

3. SINGULAR VALUES, AND NORM, INEQUALITIES

**THEOREM 3.1.** Let \( A \) be in \( C_p \), where \( 1 \leq p < \infty \), and \( B \) such that \( \| A \|^2 \geq \| B \|^2 \). Then:

(a) \( B \in C_p \) and

\[
s_i(A) \geq s_i(B) \quad \text{for} \quad i = 1, 2, \ldots;
\]

(b) if \( \text{rank } A = n < \infty \) then \( \text{rank } B \leq n \) and for every unitarily invariant norm \( \| \cdot \| : F \rightarrow \mathbb{R}^+ \)

\[
\| A \| \geq \| B \|.
\]

**Proof.** (a) **Special case.** Suppose, first, that \( \text{rank } A = n < \infty \). Then \( \| A \| = n \) and, as a consequence of (2.2),

\[
\| A \| = \sum_{i=1}^{n} \alpha_i (f_i \otimes f_i)
\]

where \( \alpha_i > 0 \), \( \| f_i \| = \alpha_i f_i \) and \( S\{f_i\} = \text{Ran } A \) \( (= \text{Ker } A^{-1}) \). As \( \| A \| \) is compact and \( \| A \|^2 \geq \| B \|^2 \) it follows from Lemma 2.1 (b) that \( \| B \| \) is compact. Therefore,

\[
\| B \| = \sum_{i=1}^{\infty} \beta_i (g_i \otimes g_i)
\]

where \( \beta_i > 0 \), \( \| g_i \| = \beta_i g_i \) and \( S\{g_i\} = \text{Ran } B \) \( (= \text{Ker } B^{-1}) \). We prove that \( \text{Ran } A \supseteq \text{Ran } B \); equivalently, \( \text{Ker } A \subseteq \text{Ker } B \). Let \( f \) be in \( \text{Ker } A \). Then, since by Lemma 2.1 (b), \( \| A \| \geq \| B \| \) we have

\[
0 \leq \langle B | f, f \rangle \leq \langle A | f, f \rangle = 0
\]

so that \( f \in \text{Ker } B \). Hence, \( \text{Ker } A \subseteq \text{Ker } B \). Therefore, \( \text{rank } B \leq n \) and so \( \text{rank } B \leq n \).
Since $|A|$ and $|B|$ are of finite rank and since $\text{Ran} \ X = (\text{Ker} \ X)^\perp$ for $X = |A|, |B|$ we can apply Löwner’s result [8, p.510, A1b] that says that if $S$ and $T$ are positive $n \times n$ matrices such that $S \geq T$ then $\lambda_i(S) \geq \lambda_i(T)$ (here, $\lambda_i(S)$ denotes the $i^{th}$ (positive) eigenvalue of $S$, the eigenvalues being arranged in decreasing order and repeated according to multiplicity). Hence, $\lambda_i(|A|) \geq \lambda_i(|B|)$, that is, $s_i(A) \geq s_i(B)$ for $i = 1, 2, \ldots$.

**General case.** We now extend this result to the von Neumann-Schatten classes.

Let now $A$ be in $C_p$, where $1 \leq p < \infty$. Then, since $|A|^2 \geq |B|^2$, it follows from Lemma 2.1 (c), (b) that $B \in C_p$ and $|A| \geq |B|$. As $A \in C_p$, then $|A| \in C_p$. Thus, $|A|-|B|$ is a positive operator in $C_p$ and so by (2.2) can be expressed as

$$|A|-|B|=\sum_{i=1}^{\infty} \gamma_i (h_i \otimes h_i)$$

where $\gamma_i > 0$, $(|A|-|B|)h_i = \gamma_i h_i$ and $S\{h_i\} = \text{Ran}(|A|-|B|)$ ($=(\text{Ker}(|A|-|B|))^\perp$). Thus, for each fixed $n (< \infty)$ each $S\{h_i\}_{i=1}^{i=n}$ reduces $|A|-|B|$. Therefore, by the second paragraph of the proof of the special case

$$s_i(A) \geq s_i(B) \quad \text{for} \quad i = 1, \ldots, n.$$  

(3.1)

To extend this to all $i = 1, 2, \ldots$ recall (cf. (2.4)) that the operator $X(=A,B)$ is the uniform limit in $\|\cdot\|_p$ of a sequence of finite rank operators $X_n$, where rank $X_n = n$. Thus (cf. (2.3))

$$X = \sum_{i=1}^{\infty} s_i(X)(l_i \otimes m_i), \quad X_n = \sum_{i=1}^{n} s_i(X)(l_i \otimes m_i)$$

so that $X - X_n = \sum_{i=n+1}^{\infty} s_i(X)(l_i \otimes m_i)$ whence, for $1 \leq p < \infty$,

$$\|X - X_n\|_p = \left[ \sum_{i=n+1}^{\infty} s_i^p(X) \right]^{\frac{1}{p}} \to 0$$

as $n \to \infty$ and so $(s_i(X))_{i=1}^{n} \to (s_i(X))^0$. Taking $X(=A,B)$ the inequality (3.1) now extends to all $i$.

(b) As in the Special case of (a), it follows that if rank $A = n < \infty$ then rank $B \leq n$. If $\phi$ is the symmetric gauge function associated, by Theorem 2.2, with the unitarily invariant norm $\|\cdot\|_\infty$ then, since $s_i(A) \geq s_i(B)$ for $i = 1, \ldots, n$, it follows from Lemma 2.4 that

$$\phi(s_1(A), \ldots, s_n(A)) \geq \phi(s_1(B), \ldots, s_n(B)),$$
that is, \( \|A\| \geq \|B\| \).

**COROLLARY 3.1** (cf. [5, Theorems 2.1, 2.2], [6, Theorem 3.3]). Let \( A \) have closed range and have a (i), (iii) inverse \( A^\dagger \) and let \( X \) be such that \( AX - C \in C_p \), where \( 1 \leq p < \infty \). Then

\[
s_i((AX - C) \geq s_i(AA^\dagger C - C) \quad \text{for} \quad i = 1, 2, \ldots \tag{3.2}
\]

and if \( AX - C \) is of finite rank so is \( AA^\dagger C - C \) and for every unitarily invariant norm \( \| \cdot \| : F \mapsto R^+ \)

\[
\|AX - C\| \geq \|AA^\dagger C - C\| \tag{3.3}
\]

with equality in (3.3) if, and for strictly convex \( \| \cdot \| \) only if, \( AX = AA^\dagger C \), that is, \( X = A^\dagger C + (I - A^\dagger A)L \) for arbitrary \( L \) in \( L(H) \); if \( A^\dagger = A^\dagger \) the least such minimizer of finite rank (with respect to \( \| \cdot \| \)) is \( A^\dagger C \).

**Proof.** As in [6, Theorem 3.3] the inequality \( |AX - C| \geq |AA^\dagger C - C| \) yields, by Theorem 3.1 (a), the inequality (3.2) and for finite rank \( AX - C \) yields, by Theorem 3.1 (b), the inequality (3.3).

Equality holds in (3.3) if \( AX = AA^\dagger C \), that is, if \( X = A^\dagger C + (I - A^\dagger A)L \) for arbitrary \( L \). If, further, \( A^\dagger = A^\dagger \) the inequality \( |A^\dagger C + (I - A^\dagger A)L| \geq |A^\dagger C| \) of [6, Theorem 3.3] yields, by Theorem 3.1 (b), that \( A^\dagger C \) is a least such minimizer with respect \( \| \cdot \| \).

For strictly convex \( \| \cdot \| \) the equality assertions follow from Lemma 2.3 and the convexity of the sets \( \{AX - C : X \text{ variable}\} \) and \( \{A^\dagger C + (I - A^\dagger A)L : L \text{ variable}\} \).

There is a similar “left-handed” result pertaining to \( XB - C \) (cf. [5, (2.4)], [6, (3.4)]). This says, amongst other things, that if \( B \) has closed range and has a (i), (iv) inverse \( B^- \) and if \( XB - C \in C_p \), where \( 1 \leq p < \infty \), then

\[
s_i((XB - C) \geq s_i((CB^-B - C) \quad \text{for} \quad i = 1, 2, \ldots \tag{3.4}
\]

It might be thought that Corollary 3.2 and (3.3) are both part of some “2-sided” result pertaining to \( AXB - C \). Certainly, it is true that if \( A \) and \( B \) have closed ranges, if \( A \) has a (i), (iii) inverse \( A^\dagger \), if \( B \) has a (i), (iv) inverse \( B^- \) and if \( AXB - C \in C_2 \) then \( AA^\dagger CB^-B - C \in C_2 \) and

\[
\|AXB - C\| \geq \|AA^\dagger CB^-B - C\|, \tag{3.5}
\]

cf [10, Corollary 1]. But it is not true that

\[
s_i((AXB - C) \geq s_i((AA^\dagger CB^-B - C) \quad \text{for} \quad i = 1, 2, \ldots \tag{3.6}
\]
EXAMPLE 3.1 [7, Example 4.1]. Let $H = R \times R$ and $A = B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ so that $A^* = B^* = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$. Take $X = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}$. Then it can be checked that $|AXB - C|^2 = \begin{pmatrix} 0 & 0 \\ 0 & 16 \end{pmatrix}$ and $|AA^*CB^*B - C|^2 = \begin{pmatrix} 2 & -2 \\ -2 & 10 \end{pmatrix}$ which gives

$$s_1(AXB - C) = 4 \geq (6 + \sqrt{20})^2 = s_1(AA^*CB^*B - C)$$

$$s_2(AXB - C) = 0 < (6 - \sqrt{20})^2 = s_2(AA^*CB^*B - C).$$

The next result contains and extends some already known inequalities [5, Theorems 3.1, 3.2], [6, Theorems 3.5, 3.6].

**THEOREM 3.2.** Let $A$ be fixed, let $P$ and $Q$ be fixed projections such that $A = PAQ$ and let $X$ vary in $C_p$, where $1 \leq p < \infty$, and be such that $A = PXQ$. Then $A \in C_p$ and

(a) \hspace{1cm} s_i(X) \geq s_i(A) \hspace{1cm} \text{for} \hspace{1cm} i = 1, 2, \ldots$

(b) \hspace{1cm} $\|X\|_p \geq \|A\|_p$

with equality occurring if, and for $1 < p < \infty$ only if, $X = A$;

(c) \hspace{1cm} $V$ is a critical point of the map $X \mapsto \|X\|_p$, for $1 < p < \infty$, if and only if $V = A$;

(d) \hspace{1cm} if $X$ is of finite rank then for every unitarily invariant norm $\| \cdot \|$, $\|X\| \geq \|A\|$ with equality occurring if, and for strictly convex $\| \cdot \|$ only if, $X = A$.

**Proof.** (a) Obviously, since $X \in C_p$ then $A (= PXQ) \in C_p$. For a projection $R$, say, $\|X\| \geq \|RX\|^2$ (equivalently, $\|Xf\| \geq \|RXf\|$ where $f \in H$). Apply Theorem 3.1 (a) successively (for the projections $P$ and $Q$)

$$s_i(X) \geq s_i(PX) = s_i(X^*P) \geq s_i(QX^*P) = s_i(PQXQ) = s_i(A) \hspace{1cm} \text{for} \hspace{1cm} i = 1, 2, \ldots$$

using the self-adjointness of $s_i(\cdot)$.

(b) The uniformity (2.5) of the $\| \cdot \|_p$ norm, for $1 \leq p < \infty$, gives, since $\|P\| = \|Q\| = 1$,

$$\|A\|_p \geq \|PXQ\|_p \leq \|P\| \|X\|_p \|Q\| = \|X\|_p.$$
For $1 < p < \infty$, the uniqueness assertion follows, by Lemma 2.2 from the convexity of the set $\{ X : PXQ = A \}$.

(c) Let $V$ be a critical point of $X \mapsto \|X\|_p$, for $1 < p < \infty$, and let $S$ be an arbitrary increment of $V$ so that $A = PVQ = P(V + S)Q$. Hence, $S \in C_p$ and $PSQ = 0$. Take $S = f \otimes g$ for vectors $f$ and $g$ in $H$. Recall that $\text{tr}[T(f \otimes g)] = \langle Tg, f \rangle$ (cf. [11, p.73, 90]). Then by Theorem 2.1

$$0 = \text{Re} \text{tr}[[V^p] U^*(f \otimes g)] = \text{Re} \langle \|V^p\|_1 U^* g, f \rangle \quad (3.5)$$

where $V = U \|V\|$ is the polar decomposition of $V$ (so that $\text{Ker} U = \text{Ker} |V|$).

As $0 = PSQ = P(f \otimes g)Q$ then $f \in \text{Ran}(I - Q)$ for arbitrary $g$ or $g \in \text{Ran}(I - P)$ for arbitrary $f$. Substituting $f = (I - Q)x$, for arbitrary $x$ in (3.5) gives $0 = \langle g, U \|V^p\|_1 (I - Q)x \rangle$ which, since $g$ is also arbitrary, forces $0 = U \|V^p\|_1 (I - Q)x$; hence, $\text{Ran} \|V^p\|_1 (I - Q) \subseteq \text{Ker} U = \text{Ker} |V|$ and so $\text{Ran}(I - Q) \subseteq \text{Ker} \|V^p\|_1 \text{Ker} U \|V\| \subseteq \text{Ker} V$. Hence, $V = VQ$. Similarly, substituting $g = (I - P)y$, for arbitrary $y$, in (3.5) yields $V = PV$.

Thus: $V = PV = PVQ = A$ as desired.

Conversely, by (b), $A$ is a global minimizer of $X \mapsto \|X\|_p$ and hence, for $1 < p < \infty$, is a critical point of it.

(d) As in (b), the inequality follows from the uniformity (2.7) of the $\| \cdot \|$ norm and the uniqueness assertion follows, via Lemma 2.3, from the convexity of the set $\{ X : PXQ = A \}$. (Alternatively, the inequality follows, via Theorem 3.1 (b), from (a).) ■

There is no version of Theorem 3.2 (c) for $0 < p < 1$ in finite dimensions since if the critical point $V$ and hence, by the proof of Theorem 3.2 (c), $A (= V)$ were invertible, and hence $1 - 1$ and onto, the condition $A = PAQ$ would force $P = Q = I$ and hence $X = A$ so that $\| X \|_p^p$ would be constant.

For $P \neq I$ or $Q \neq I$ the hypotheses of Theorem 3.2 that $A = PXQ$ and $X \in C_p$, where $1 \leq p < \infty$, force $A$ and $A^*$ to be finite rank. For suppose $P \neq I$; since $A = PXQ \in C_p$ then $A$ is compact and has closed range and so is of finite rank. As $A$ has closed range it has the Moore-Penrose inverse $A^*$; hence $A^* = (AA^* A)^{-1} A^* (AA^*)$ (by (2.1) (iii)) so $A^*$ is also of finite rank (Of course, if $Q \neq I$ this is obvious from $A^* = QX^* P \in C_p$). Hence, in Corollary (3.2), $A$ and $A^*$ are of finite rank. In Corollaries 3.3 and 3.4 the operator $A^*$, being compact and
having closed range, is of finite rank; hence, the operators $A = AA^*A$ and $A^* = A^*AA^*$ are of finite rank.

**COROLLARY 3.2** (cf. [6, Theorem 3.5]). Let the operators $A$ and $B$ be fixed, let $B$ have closed range and have a (i), (iv) inverse $B^*$ and satisfy $A = B^*BA$ and let $X$ vary in $C_p$, where $1 < p < \infty$, and satisfy $A = B^*BX$. Then $A \in C_p$ and

$$s_i(X) \geq s_i(A) \quad \text{for } i = 1, 2, \ldots$$

**Proof.** Immediate from Theorem 3.2 (a) on taking the projection $P = B^*B$ and $Q = I$.

**COROLLARY 3.3** (cf. [6, Theorem 3.6]). Let $A$ have closed range and have the Moore-Penrose inverse $A^*$ and let $X$ vary in $C_p$, where $1 < p < \infty$, and $A^*AX = A^*$. Then $A^* \in C_p$ and

$$s_i(X) \geq s_i(A^*) \quad \text{for } i = 1, 2, \ldots$$

**Proof.** As shown in [6, Theorem 3.6] the hypothesis $A^*AX = A^*$ is equivalent to $A^*AX = A^*$. The result is now immediate from Theorem 3.2 (a) on taking $P = A^*A$, $Q = I$ and $A^* = A^*$.

**COROLLARY 3.4** (cf. [5, Theorem 3.1]). If $A$ has a generalized inverse $A^-$ in $C_p$, where $1 < p < \infty$, then the Moore-Penrose inverse $A^+ \in C_p$ and

$$s_i(A^-) \geq s_i(A^+) \quad \text{for } i = 1, 2, \ldots$$

**Proof.** By [5, Lemma 1.4] $A^+ = PA^+Q$ where $P$ and $Q$ are the projections onto $(\ker A)^\perp$ and $\ran A$ respectively. On taking $A^* = A^*$ and $X^* = A^*$ the result is immediate from Theorem 3.2 (a).

Obviously, to the $s_i(\cdot)$ inequalities of Corollaries 3.2, 3.3 and 3.4 there are, by Theorem 3.2 (d), corresponding inequalities in $||| \cdot |||$ (and Theorem 3.2 (b), (c) implies already known inequalities in $|| \cdot ||_p$ for $1 \leq p \leq \infty$ and local results pertaining to $X \mapsto ||X||_p$ for $1 < p < \infty$; the $p = \infty$ inequality follows since if $A = PAQ = PXQ$ then $||X|| \geq ||A||$).

**REFERENCES.**

Address:
Mathematics and Statistics Group, Middlesex University, Hendon Campus, The Burroughs, LONDON NW4 4BT, ENGLAND

E-mail: p.maher@mdx.ac.uk