COMPOSITION OPERATORS FROM BLOCH TYPE SPACES TO $F(p, q, s)$ SPACES

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Abstract

Suppose that $\varphi$ is an analytic self-map of the unit disk, the compactness of the composition operator $C_\varphi$ from the Bloch type space into the space $F(p, q, s)$ is investigated.

1 Introduction

Let $D$ be the open unit disk in the complex plane $\mathbb{C}$ and $\partial D$ the unit circle. Denote by $H(D)$ the class of all functions analytic on $D$. An $f \in H(D)$ is said to belong to the Bloch type space, or $\alpha$--Bloch space $B^\alpha$ if

$$B(f) = \sup_{z \in D} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$  

Note that $B^\alpha$ is a Banach space with the norm which is given by $\|f\|_{B^\alpha} = |f(0)| + B(f)$. When $\alpha = 1$, $B = B^1$ is the well known Bloch space. Let $B^\alpha_0$, called the little Bloch type space, denote the subspace of $B^\alpha$ consisting of those $f \in B^\alpha$ for which $\lim_{|z| \to 1} (1 - |z|^2)^\alpha |f'(z)| = 0$.

The one-to-one holomorphic functions that map $D$ onto itself have the form $\lambda \varphi_a$, where $\lambda \in \partial D$ and $\varphi_a$ is the basic conformal automorphism

2000 Mathematics Subject Classification. Primary 47B35, Secondary 30H05.

Key words and phrases. composition operator, Bloch type spaces, $F(p, q, s)$ spaces

Received: April 11, 2006
defined by $\varphi_a = \frac{a - z}{1 - \overline{a}z}$ for $a \in D$. It is easy to check that the following equalities hold

$$\varphi_a \circ \varphi_a(z) = z, \quad |\varphi_a'(z)| = \frac{1 - |a|^2}{|1 - \overline{a}z|^2}, \quad 1 - |\varphi_a(z)|^2 = (1 - |z|^2)|\varphi_a'(z)|^2.$$  

For $a \in D$, let $g(z, a)$ be Green’s function for $D$ with logarithmic singularity at $a$, i.e. $g(z, a) = \log \frac{1}{|\varphi_a(z)|}$. Let $0 < p, s < \infty$, $-2 < q < \infty$. A function $f \in H(D)$ is said to belong to $F(p, q, s)$ (see [14]) if

$$\|f\|_{F(p, q, s)}^p = \sup_{a \in D} \int_D |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < \infty,$$

and $f \in F_0(p, q, s)$ if $f \in H(D)$ and

$$\lim_{|a| \to 1} \int_D |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) = 0.$$  

$F(p, q, s)$ is a Banach space under the norm $\|f\|_{F(p, q, s)}^p = |f(0)| + \|f\|_{p, q, s}^p$. $F(p, q, s)$ is called general function space because it can get many function spaces if it takes special parameters of $p, q, s$. For example, $F(p, q, s) = B^{q+s}_{\frac{q+s}{p}}$ and $F_0(p, q, s) = B^{q+s}_{\frac{q+s}{p}}$ for $s > 1$; $F(p, q, s) \subset B^{q+s}_{\frac{q+s}{p}}$ and $F_0(p, q, s) \subset B^{q+s}_{\frac{q+s}{p}}$ for $0 < s \leq 1$; $F(2, 0, s) = Q_s$ and $F_0(2, 0, s) = Q_{s,0}$; $F(2, 0, 1) = BMOA$ and $F_0(2, 0, 1) = VMOA$; If $q + s \leq -1$, then $F(p, q, s)$ is the space of constant functions.

Let $\varphi$ be an analytic self-map of $D$. Then the composition operator $C_\varphi$ with symbol $\varphi$ is defined by

$$C_\varphi f = f \circ \varphi$$

for $f \in H(D)$. Littlewood’s subordination principle gives that $C_\varphi$ is a bounded linear operator on the classical Hardy and Bergman spaces. More information about the study of composition operators can be found in [2, 16].

In [15], Zhao has characterized the boundedness and compactness of composition operators between the Bloch type spaces and the Hardy and Besov spaces. Smith and Zhao have characterized the boundedness of $C_\varphi : B \to Q_p$, $C_\varphi : B_0 \to Q_{p,0}$ and $C_\varphi : B \to Q_{p,0}$ in [9]. In [11], Wulan has characterized the compactness of composition operators between the
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Bloch space and the $Q_K$ space. Some related results can be founded in [1, 4, 5, 6, 8, 12].

In this paper we study the composition operators from the Bloch type space $B^\alpha$ into the space $F(p, q, s)$. For a subarc $I \in \partial D$, let

$$S(I) = \{r\zeta \in D : 1 - |I| < r < 1, \zeta \in I\}.$$ 

If $|I| \geq 1$, then we set $S(I) = D$. For $r \in (0, 1)$, let $D_r = \{z \in D : |\varphi(z)| > r\}$. The characteristic function of a set $E \subset D$ is denoted by $\mathbb{1}_E$. Jiang and He in [3] studied the boundedness and compactness of composition operator from the Bloch type space $B^\alpha$ into the space $F(p, q, s)$. The main results in [3] can be stated as follows.

**Theorem A.** Let $\varphi$ be an analytic self-map of $D$, $0 < \alpha, p, s < \infty$, $-2 < q < \infty$ and $q + s > -1$. The following statements are equivalent:

1. $C_\varphi : B^\alpha \to F(p, q, s)$ is bounded;
2. For $p \geq 2$, $C_\varphi : B^\alpha_0 \to F(p, q, s)$ is bounded;
3. 

$$\sup_{a \in D} \int_D \frac{|\varphi'(z)|^p}{(1 - |\varphi(z)|^2)^{p\alpha}(1 - |z|^2)^q g^s(z, a)} dA(z) < \infty. \quad (1)$$

**Theorem B.** Let $\varphi$ be an analytic self-map of $D$, $0 < \alpha, p, s < \infty$, $-2 < q < \infty$ and $q + s > -1$. The following statements are equivalent:

1. $C_\varphi : B^\alpha \to F(p, q, s)$ is a compact operator;
2. $C_\varphi : B^\alpha_0 \to F(p, q, s)$ is a compact operator;
3. $\varphi \in F(p, q, s)$ and

$$\lim_{r \to 1} \sup_{I \subset \partial D} |I|^{-s} \int_{S(I)} I_{D_r} \frac{|\varphi'(z)|^p}{(1 - |\varphi(z)|^2)^{p\alpha}(1 - |z|^2)^{q+s}} dA(z) = 0. \quad (2)$$

The above compactness condition is very difficult to verify. In this paper, we give another characterization of the compactness of $C_\varphi$ from Bloch type space $B^\alpha$ into the space $F(p, q, s)$.

Throughout this paper, $C$ always denote positive constant and may be different at different occurrences.
2 Main Results and Proofs

In this section, we give the main results and the proofs of this paper by using the methods of [11]. For this purpose, we need some lemmas. The following criterion for compactness follows by standard arguments similar, for example, to those outlined in Proposition 3.11 of [2].

**Lemma 1.** Let \( \varphi \) be an analytic self-map of \( D, 0 < \alpha, p, s < \infty, -2 < q < \infty \) and \( q + s > -1 \). Then \( C_{\varphi} : B^\alpha \to F(p,q,s) \) is a compact operator if and only if \( C_{\varphi} : B^\alpha \to F(p,q,s) \) is bounded and for any bounded sequence \( (f_n) \) in \( B^\alpha \) with \( (f_n) \to 0 \) uniformly on compact sets as \( n \to \infty \), \( \|C_{\varphi}f_n\|_{F(p,q,s)} \to 0 \), as \( n \to \infty \).

**Lemma 2.** Let \( \varphi \) be an analytic self-map of \( D, 0 < \alpha, p, s < \infty, -2 < q < \infty \) and \( q + s > -1 \). If \( C_{\varphi} : B^\alpha(B_0^\alpha) \to F(p,q,s) \) is a compact operator, then for any \( \varepsilon > 0 \) there exists a \( \delta, 0 < \delta < 1 \), such that for all \( f \) in \( B_{B_0^\alpha}(B_{B_0^\alpha}) \), the unit ball of \( B^\alpha(B_0^\alpha) \),

\[
\sup_{a \in D} \int_{|\varphi(z)| > r} |(f \circ \varphi)'(z)|^p(1 - |z|^2)^q g^s(z,a) dA(z) < \varepsilon
\]

holds whenever \( \delta < r < 1 \).

**Proof.** We only prove the case of \( B_0^\alpha \). The proof for \( B^\alpha \) is similar, hence we omit the details. Assume that \( C_{\varphi} : B_0^\alpha \to F(p,q,s) \) is compact. For \( f \in B_{B_0^\alpha} \), let \( f_t(z) = f(tz) \) for \( t \in (0,1) \) and \( z \in D \). Then \( f_t \to f \) uniformly on compact subsets of \( D \) as \( t \to 1 \). Since \( C_{\varphi} \) is compact, then by Lemma 1 we see that \( \|C_{\varphi}f_t - C_{\varphi}f\|_{p,q,s} \to 0 \) as \( t \to 1 \). Thus, for given \( \varepsilon > 0 \), there is a \( t \in (0,1) \) such that

\[
\sup_{a \in D} \int_D |f'_t(\varphi(z)) - f'(\varphi(z))|^p |\varphi'(z)|^p(1 - |z|^2)^q g^s(z,a) dA(z) < \varepsilon.
\]
By the triangle inequality, for \( r \in (0, 1) \), we have
\[
\sup_{a \in D} \int_{|\varphi(z)| > r} |f'(\varphi(z))|^p |\varphi'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z)
\leq \sup_{a \in D} \int_{|\varphi(z)| > r} |f'(\varphi(z)) - f'(\phi(z))|^p |\varphi'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z)
+ \sup_{a \in D} \int_{|\varphi(z)| > r} |f'(\varphi(z))|^p |\varphi'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z)
\leq \varepsilon + \|f'_1\|_{\infty}^p \sup_{a \in D} \int_{|\varphi(z)| > r} |\varphi'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z).
\]

Now, we prove that for given \( \varepsilon > 0 \) and \( \|f'_1\|_{\infty}^p > 0 \) there exists a \( \delta \in (0, 1) \) such that if \( \delta < r < 1 \)
\[
\|f'_1\|_{\infty}^p \sup_{a \in D} \int_{|\varphi(z)| > r} |\varphi'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < \varepsilon.
\]

Let \( f_n(z) = n^{a-1}z^n \). It is easy to see that \( f_n \in B^a_0 \) and converges to zero uniformly on compact subsets of \( D \). Since \( C_\varphi \) is a compact operator, we have \( \lim_{n \to \infty} \|n^{a-1}\varphi^n\|_{p, q, s} \to 0 \) as \( n \to \infty \). That is, for any given \( \varepsilon > 0 \) and \( \|f'_1\|_{\infty}^p > 0 \), there exists a integer \( N > 1 \) such that
\[
\|f'_1\|_{\infty}^p \sup_{a \in D} \int_{|\varphi(z)| > r} n^{p a} |\varphi'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < \varepsilon,
\]
whenever \( n \geq N \). Given \( r \in (0, 1) \), (4) yields
\[
N^{ap} r^{pN - p} \int_{|\varphi(z)| > r} |\varphi'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < \varepsilon.
\]
Taking \( r = N^{-\frac{a}{N-p}} \), we get
\[
\|f'_1\|_{\infty}^p \sup_{a \in D} \int_{|\varphi(z)| > r} |\varphi'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < \varepsilon.
\]

Hence we have already proved that for any \( \varepsilon > 0 \) and for \( f \in B_{B^a_0} \), there exists a \( \delta = \delta(\varepsilon, f) \) such that
\[
\sup_{a \in D} \int_{|\varphi(z)| > r} |(f \circ \varphi)'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < \varepsilon
\]
holds whenever \( \delta < r < 1 \).
We finish the proof of Lemma 2 by showing that the \( \delta = \delta(\varepsilon, f) \), in fact, is independent of \( f \in \mathcal{B}_{\mathcal{B}_0^\alpha} \). Since \( C_\varphi(\mathcal{B}_{\mathcal{B}_0^\alpha}) \) is a relatively compact subset of \( F(p, q, s) \), there are a finite collection of functions \( f_1, f_2, \cdots, f_n \) in \( \mathcal{B}_{\mathcal{B}_0^\alpha} \) such that for any \( \varepsilon > 0 \) and \( f \in \mathcal{B}_{\mathcal{B}_0^\alpha} \), there is a \( k = 1, 2, \cdots, n \), satisfying

\[
\sup_{a \in D} \int_D |f'(\varphi(z)) - f_k'(\varphi(z))|^p |\varphi'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < \varepsilon.
\]

On the other hand, if \( \max_{1 \leq k \leq n} \delta(\varepsilon, f_k) = \delta < r < 1 \), we have for all \( k = 1, 2, \cdots, n \),

\[
\sup_{a \in D} \int_{|\varphi(z)| > r} |f_k'(\varphi(z))|^p |\varphi'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < \varepsilon.
\]

By using the triangle inequality, we get

\[
\sup_{a \in D} \int_{|\varphi(z)| > r} |(f \circ \varphi)'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < \varepsilon
\]

whenever \( \delta < r < 1 \). The proof is completed.

**Lemma 3.**[17] Suppose that \( n_k \) is an increasing sequence of positive integers with Hadamard gaps, that is, \( n_{k+1}/n_k \geq \lambda > 1 \) for all \( k \). Let \( 0 < p < \infty \). Then there is a constant \( M > 0 \) depending on \( p \) and \( \lambda \) such that

\[
M^{-1} \left( \sum_{k=1}^N |a_k|^2 \right)^{1/2} \leq \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^N a_k e^{i n_k \theta} \right|^p d\theta \right)^{1/p} \leq M \left( \sum_{k=1}^N |a_k|^2 \right)^{1/2}
\]

for any scalars \( a_1, \ldots, a_N \) and \( N = 1, 2, \ldots \).

We are now ready to state and prove the main results in this section.

**Theorem 1.** Let \( \varphi \) be an analytic self-map of \( D \), \( 0 < p, \alpha, s < \infty \), \( -2 < q < \infty \) and \( p + s > -1 \). The following statements are equivalent:

(i) \( C_\varphi : \mathcal{B}^\alpha \to F(p, q, s) \) is a compact operator;

(ii) \( C_\varphi : \mathcal{B}_{\mathcal{B}_0^\alpha} \to F(p, q, s) \) is a compact operator;

(iii) \( \varphi \in F(p, q, s) \) and

\[
\lim_{r \to 1} \sup_{a \in D} \int_{|\varphi(z)| > r} \frac{|\varphi'(z)|^p}{(1 - |\varphi(z)|^2)^{p\alpha}} (1 - |z|^2)^q g^s(z, a) dA(z) = 0.
\]
Proof. (i) ⇒ (ii) It is trivial since \( B^0_0 \subset B^\alpha \).

(ii) ⇒ (iii) Suppose that \( C_\varphi : B^\alpha_0 \to F(p, q, s) \) is compact. By choosing \( f = z \in B^0_0 \) we have \( \varphi \in F(p, q, s) \). Next, we choose the function

\[
f(z) = \sum_{k=1}^{\infty} 2^{k(\alpha-1)} z^{2^k},
\]

we see that \( f(z) \in B^\alpha \) from \([13]\). Set \( g(z) = f(z)/\|f\|_{B^\alpha} \), choose a sequence \( \{\lambda_n\} \) in \( D \) which converges to 1 as \( n \to \infty \), and let \( g_n(z) = g(\lambda_n z) \) for all \( n \in \mathbb{N} \). For \( 0 \leq \theta \leq 2\pi \), set \( g_{n, \theta}(z) = g_n(e^{i\theta} z) \). It is easy to see that \( g_{n, \theta} \in B^\alpha \). Replace \( f \) by \( g_{n, \theta} \) in Lemma 2 and then integrate against \( d\theta \), by Fubini’s Theorem and Lemma 3 we obtain

\[
\varepsilon \geq \frac{1}{2\pi} \int_{|\varphi(z)|>r} \left( \int_0^{2\pi} |g'_n(\xi^{i\theta} \varphi(z))|^p d\theta \right) |\varphi'(z)|^p (1 - |z|^2)^q g_s(z, a) dA(z)
\]

\[
\geq C \int_{|\varphi(z)|>r} \left( \sum_{k=1}^{\infty} 2^{2\alpha k} |\lambda_n \varphi(z)|^{2(2^k - 1)} \right)^{p/2} |\lambda_n \varphi'(z)|^p (1 - |z|^2)^q g_s(z, a) dA(z).
\]

(4)

Notice that (see \([3]\) or \([15]\))

\[
\sum_{k=1}^{\infty} 2^{2\alpha k} |\lambda_n \varphi(z)|^{2(2^k - 1)} > \frac{C(\alpha)}{(1 - |\lambda_n \varphi(z)|^2)^{2\alpha}}.
\]

(5)

Here \( C(\alpha) \) is only depend on \( \alpha \). Therefore, for \( \delta < r < 1 \) and for sufficient large \( n \), (4) and (5) give

\[
\sup_{a \in D} \int_{|\varphi(z)|>r} \frac{|\lambda_n \varphi'(z)|^p}{(1 - |\lambda_n \varphi(z)|^2)^{p\alpha}} (1 - |z|^2)^q g_s(z, a) dA(z) < C\varepsilon.
\]

By Fatou’s Lemma we obtain

\[
\lim_{r \to 1} \sup_{a \in D} \int_{|\varphi(z)|>r} \frac{|\varphi'(z)|^p}{(1 - |\varphi(z)|^2)^{p\alpha}} (1 - |z|^2)^q g_s(z, a) dA(z) = 0.
\]

(iii) ⇒ (i) Suppose that \( \varphi \in F(p, q, s) \) and (3) holds. Then it is easy to check that \( C_\varphi : B^\alpha \to F(p, q, s) \) is bounded. Let \( \{f_n\} \subset B^0_0 \). We only
need to show that \( \{C \varphi f_n\} \) has a subsequence that converges in \( F(p, q, s) \). Since \( B_{\mathcal{B}_0} \) is a normal family, by passing to a subsequence, we may assume, without loss of generality, that \( \{f_n\} \) converges to 0 uniformly on compact subsets of \( D \). By the Cauchy’s estimate, we see that \( \{f_n\}' \) also converges to 0 uniformly on compact subsets of \( D \). We must show that \( \{C \varphi f_n\} \) converges to 0 in the topology of the norm of \( \| \cdot \|_{F\left(p, q, s\right)} \). Given \( \varepsilon \in (0, 1) \), by (3), there is an \( r \in (0, 1) \) such that for all the functions \( f_n \) and all \( a \in D \),

\[
\int_{|\varphi(z)|>r} |f_n'(\varphi(z))\varphi'(z)|^p(1 - |z|^2)^q g^s(z, a)dA(z) < \varepsilon. \tag{6}
\]

Since \( D_r = \{z \in D : |z| \leq r\} \) is a compact subset of \( D \), \( \{f_n\}' \) also converges to 0 uniformly on \( D_r \). Therefore, there exists an integer \( N > 1 \) such that as \( n \geq N \),

\[
\int_{|\varphi(z)|\leq r} |f_n'(\varphi(z))\varphi'(z)|^p(1 - |z|^2)^q g^s(z, a)dA(z) < \varepsilon \|\varphi\|_{p, q, s}^p. \tag{7}
\]

Therefore, by (6) and (7),

\[
\int_{D} |f_n'(\varphi(z))\varphi'(z)|^p(1 - |z|^2)^q g^s(z, a)dA(z) < \varepsilon (1 + \|\varphi\|_{p, q, s}^2)
\]

when \( n \geq N \). That is, \( \|C \varphi f_n\|_{p, q, s} \to 0 \) as \( n \to \infty \). Therefore \( \|C \varphi f_n\|_{F\left(p, q, s\right)} \to 0 \) as \( n \to \infty \). By Lemma 1, we see that \( C \varphi : \mathcal{B}^\alpha \to F(p, q, s) \) is a compact operator.

**Corollary 1.** Let \( \varphi \) be an analytic self-map of \( D \). Then the following statements are equivalent:

(i) \( C \varphi : \mathcal{B} \to \mathcal{B} \) is a compact operator;

(ii) \( C \varphi : \mathcal{B}_0 \to \mathcal{B} \) is a compact operator;

(iii) \( \varphi \in \mathcal{B} \) and

\[
\limsup_{r \to 1} \sup_{a \in D} \int_{|\varphi(z)|>r} \frac{|\varphi'(z)|^p}{(1 - |\varphi(z)|^2)^p} (1 - |z|^2)^{p-2} g^s(z, a)dA(z) = 0
\]

for all \( p > 0 \) and all \( s > 1 \);  

(iv) \( \varphi \in \mathcal{B} \) and

\[
\limsup_{r \to 1} \sup_{a \in D} \int_{|\varphi(z)|>r} \frac{|\varphi'(z)|^p}{(1 - |\varphi(z)|^2)^p} (1 - |z|^2)^{p-2} g^s(z, a)dA(z) = 0
\]
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for each $p > 0$ and each $s > 1$.

Proof. Since $\mathcal{B} = F(p, p-2, s)$ for any $p > 0$ and any $s > 1$ (see Theorem 1.3 in [14]), the result is a direct consequence of Theorem 1.

Remark 1. The compactness of composition operator on Bloch space was characterized in [7]. In [10], Tjani proved that $C_\varphi : \mathcal{B} \to \mathcal{B}$ is compact if and only if $\lim_{|a| \to 1} \|C_\varphi \varphi_a\|_{\mathcal{B}} = 0$. Another related result can be found in [11].

Remark 2. From the proof of Theorem 1 and the proof of Theorem 1.1 in [3], we find that by using Lemma 3, we can remove the restrict condition $p \geq 2$ in Theorem A, i.e. $p \geq 2$ in Theorem 1.1 in [3].

References


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