A GENERALISED COMMUTATIVITY THEOREM FOR $PK$-QUASIHYPONORMAL OPERATORS

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Abstract

For Hilbert space operators $A$ and $B$, let $\delta_{AB}$ denote the generalised derivation $\delta_{AB}(X) = AX - XB$ and let $\triangle_{AB}$ denote the elementary operator $\triangle_{AB}(X) = AXB - X$. If $A$ is a $pk$-quasihyponormal operator, $A \in pk^{}-QH$, and $B^*$ is an either $p$-hyponormal or injective dominant or injective $pk^{}-QH$ operator (resp., $B^*$ is an either $p$-hyponormal or dominant or $pk^{}-QH$ operator), then $\delta_{AB}(X) = 0 \implies \delta_{A^*B^*}(X) = 0$ (resp., $\triangle_{AB}(X) = 0 \implies \triangle_{A^*B^*}(X) = 0$).

1 Introduction

Let $B(\mathcal{H}, \mathcal{K})$, $B(\mathcal{H}) = B(\mathcal{H}, \mathcal{H})$, denote the algebra of operators (equivalently, bounded linear transformations) from a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$. Let $\delta_{AB} \in B(B(\mathcal{K}), B(\mathcal{H}))$, $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$, denote the generalised derivation $\delta_{AB}(X) = AX - XB$, and let $\triangle_{AB} \in B(B(\mathcal{K}), B(\mathcal{H}))$ denote the elementary operator $\triangle_{AB}(X) = AXB - X$. The (classical) Putnam-Fuglede commutativity theorem says that if $A$ and $B$ are normal operators, then $\delta_{AB}^{-1}(0) \subseteq \delta_{A^*B^*}^{-1}(0)$. Over the years, the Putnam-Fuglede commutativity theorem has been extended to various classes of operators, each more general than the class of normal operators, and to the elementary operator $\triangle_{AB}$ to prove that $\triangle_{AB}^{-1}(0) \subseteq \triangle_{A^*B^*}^{-1}(0)$ for many of these classes of operators (see [1, 2, 3, 7] and [9] for references). Recall that an operator $A \in B(\mathcal{H})$ is said to be $(p, k)$-quasihyponormal, $A \in pk^{}-QH$, for some real number $0 < p \leq 1$ and non-negative integer $k$ (momentarily, we allow $k = 0$) if $A^{*k}(|A|^{2p} - |A^*|^{2p})A^k \geq 0$. Evidently, a $10^{}-QH$ operator is hyponormal.
a $p0 - QH$ operator is $p$-hyponormal, a $11 - QH$ operator is quasihyponormal and a $1k - QH$ operator for $k \geq 1$ is $k$-quasihyponormal. Recently, Kim [9, Theorem 11] has proved that if $A \in B(\mathcal{H})$ is an injective $pk - QH$ operator, $k \geq 1$, and $B^* \in B(\mathcal{K})$ is a $p$-hyponormal operator, then $\delta^{-1}_{AB}(0) \subseteq \delta^{-1}_{A^*B^*}(0)$. Using what is essentially a very simple argument, we prove in this note a more general result which not only leads to Kim’s result (loc.cit.) but also gives us further similar results. Thus we prove that if $A \in pk - QH$ and $B^*$ is either $p$-hyponormal or injective dominant or injective $pk - QH$, then $\delta^{-1}_{AB}(0) \subseteq \delta^{-1}_{A^*B^*}(0)$, and if $A \in pk - QH$ and $B^*$ is either $p$-hyponormal or dominant or $pk - QH$, then $\Delta^{-1}_{AB}(0) \subseteq \Delta^{-1}_{A^*B^*}(0)$. We also consider operators $A \in pk - QH$ for which $\delta_{A^*A}(X) = 0$ or $\Delta_{A^*A}(X) = 0$ for some invertible operator $X$.

In the following we shall denote the closure of a set $S$ by $\overline{S}$, the range of $T \in B(\mathcal{H})$ by $\text{ran} T$, the orthogonal complement of $T^{-1}(0)$ by $\ker T$, the spectrum of $T$ by $\sigma(T)$, the point spectrum of $T$ by $\sigma_p(T)$, and the class of $p$-hyponormal operators, $0 < p < 1$, by $p - \mathcal{H}$. Recall that an operator $T$ is a quasiaffinity if it is injective and has dense range. Any other notation or terminology will be defined at the first instance of its occurrence.

2 Results

Let $\mathcal{P}_1$ denote the class of operators $A \in B(\mathcal{H})$ such that

$$
A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},
$$

where $A_{22}^k = 0$ for some integer $k \geq 1$, and let $\mathcal{P}_2$ denote the class of operators $B \in B(\mathcal{K})$ such that $B$ has the decomposition $B = B_n \oplus B_p$ into its normal and pure (= completely non-normal) parts, with respect to some decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, such that $B_p$ has dense range. Let $(\mathcal{P}_1, \mathcal{P}_2)$ (resp., $(\mathcal{P}_1, \mathcal{P}_2)$) denote the class of operators $(A, B)$, $A \in \mathcal{P}_1$ and $B \in \mathcal{P}_2$, such that $\delta^{-1}_{A_{11}B_p}(0) \subseteq \delta^{-1}_{A_{11}B_p}(0)$ and $\delta^{-1}_{A_{11}B_n}(0) \subseteq \delta^{-1}_{A_{11}B_n}(0)$ (resp., $\Delta^{-1}_{A_{11}B_n}(0) \subseteq \Delta^{-1}_{A_{11}B_n}(0)$.) The following theorem is our main result.

**Theorem 2.1.** (i) If $(A, B) \in (\mathcal{P}_1, \mathcal{P}_2)$ and $X \in B(\mathcal{K}, \mathcal{H})$ is a quasiaffinity, then $X \in \delta^{-1}_{AB}(0) \implies X \in \delta^{-1}_{A^*B^*}(0)$.

(ii) If $(A, B) \in [\mathcal{P}_1, \mathcal{P}_2]$ and $X \in B(\mathcal{K}, \mathcal{H})$ is a quasiaffinity, then $X \in \Delta^{-1}_{AB}(0) \implies X \in \Delta^{-1}_{A^*B^*}(0)$. 
Proof. Let the quasiaffinity $X : K_1 \oplus K_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$ have the matrix representation $X = [X_{ij}]^2_{i,j=1}$.

(i) If $X \in \delta^{-1}_{A_p}(0)$, then $\delta_{A_{22}B_p}(X_{22}) = 0 \implies X_{22}B_p^* = 0 \implies X_{22} = 0$, since $B_p$ has dense range. Evidently, the hypothesis $\delta^{-1}_{A_{11}B_p}(0) \subseteq \delta^{-1}_{A_{11}^*A_p}(0) \implies \delta_{A_{11}B_p}(X_{12}) = 0 = \delta_{A_{11}^*A_p}(X_{12}) \implies \text{ran}X_{12}$ reduces $A_{11}$, $\ker^\perp X_{12}$ reduces $B_p$, and $A_{11}|_{\text{ran}X_{12}}$ and $B_p|_{\ker^\perp X_{12}}$ are unitarily equivalent normal operators. Hence $X_{12} = 0$, which since $X$ is a quasiaffinity implies that $K_2 = \mathcal{H}_2 = \{0\}$. Consequently, the hypothesis $\delta^{-1}_{A_{11}B_p}(0) \subseteq \delta^{-1}_{A_{11}^*A_p}(0)$ implies that $\delta_{A_{11}B_p}(X) = 0$.

(ii) If $\triangle_{AB}(X) = 0$, then

$$\triangle_{A_{22}B_p}(X_{21}) = 0 \implies X_{21} = A_{22}X_{21}B_p = A_{22}X_{21}B_n^* = 0,$$

and

$$\triangle_{A_{22}B_p}(X_{22}) = 0 \implies X_{22} = A_{22}X_{22}B_p = A_{22}X_{22}B_n^* = 0.$$ 

Since $X$ is a quasiaffinity, $K_2 = \mathcal{H}_2 = \{0\}$. Consequently, the hypothesis $\triangle^{-1}_{A_{11}B_n}(0) \subseteq \triangle^{-1}_{A_{11}^*A_p}(0)$ implies that $\triangle_{A_{11}B_p}(X) = 0$. □

The numerical range $W(T)$ of an operator $T \in B(\mathcal{H})$ is the set $\{\langle Tx, x \rangle : ||x|| = 1\}$. Recall from Embry [6] that if $A$ and $B \in B(\mathcal{H})$ are commuting normal operators, and if $X \in B(\mathcal{H})$ is such that $0 \notin W(X)$ and $\delta_{AB}(X) = 0$, then $A = B$. Thus, if $A$ is a normal operator such that $\delta_{A^*A}(X) = 0$ for some operator $X$ such that $0 \notin W(X)$, then $A$ is self-adjoint. That a similar result holds for operators $A \in \mathcal{P}_1$ is proved in [9, Theorem 2]. In the following we prove an analogue of Embry’s result for operators $A \in \mathcal{P}_1$ such that $\delta_{A^*A}(X) = 0$ or $\triangle_{A^*A}(X) = 0$. Let $\partial \mathcal{D}$ denote the boundary of the unit disc.

Theorem 2.2. Let $A \in \mathcal{P}_1$ have the decomposition $A = A_n \oplus A_p$ into its normal and pure parts (alongwith the matrix decomposition above). Let $X \in B(\mathcal{H})$ be invertible.

(i) If $0 \notin W(X)$, $\delta_{A^*A}(X) = 0$ and $\delta^{-1}_{A_{11}^*A_p}(0) \subseteq \delta^{-1}_{A_{11}A_p}(0)$, then $A$ is the direct sum of a self-adjoint operator and a nilpotent operator (where either component may act on the trivial space.)

(ii) If $\triangle_{A^*A}(X) = 0$ and $\triangle^{-1}_{A_{11}^*A_p}(0) \subseteq \triangle^{-1}_{A_{11}A_p}(0)$, then $A$ is unitary.

Proof. (i) If we let $A^*$ have the matrix representation above, $A = A_n \oplus A_p$ and $X = [X_{ij}]^2_{i,j=1}$, then $\delta_{A_{11}^*A_p}(X_{12}) = 0$. Hence, since $\delta^{-1}_{A_{11}^*A_p}(0) \subseteq \delta^{-1}_{A_{11}A_p}(0)$, $\ker^\perp(X_{12})$ reduces $A_p$, and $A_{11}|_{\text{ran}(X_{12})}$ and $A_p|_{\ker^\perp X_{12}}$ are unitarily equivalent normal operators [3, Lemma 1(i)]. Since $A_p$ is pure, we must...
have that $X_{12} = 0$. Consequently, $X_{22}$ is injective. Since $X_{22} \in \delta_{A_{22}}^{-1}(A_p)$ and $A_{22}^k = 0$, $X_{22}A_p^k = 0 \implies A_p$ is $k$-nilpotent. To complete the proof we observe now that if $A^* = XAX^{-1}$ and $0 \notin \overline{W(X)}$, then $\sigma(A)$ is real [9, Lemma 3]. Since $\sigma(A) = \sigma(A_n) \cup \sigma(A_p)$ and $A_n$ is normal, $A_n$ is self-adjoint.

(ii) Representing $A^*$, $A$ and $X$ as in (i) above, it is seen that

$$\triangle_{A_{11}, A_p}(X_{12}) = 0 = \triangle_{A_{11}, A_p^*}(X_{12}),$$

and hence that $\ker^+(X_{12})$ reduces $A_p$ and $A_p|\ker^+(X_{12})$ is normal [3, Lemma 1(ii)]. Since $A_p$ is pure, $X_{12} = 0$ and $X_{22} \in \triangle_{A_{22}}^{-1}(A_p)$. Since $A_{22}$ is $k$-nilpotent, $X_{22} = 0 \implies X = X_{11}$ and $A = A_{n}$ is normal. Hence $A^*XA = X = AXA^*$ [3, Corollary 3] $\implies |X|^2A = (AX^*A^*)XA = (AX^*)A^*XA = A|X|^2 \implies |X|A = A|X|$. Letting $X$ have the polar decomposition $X = U|X|$, it follows that $A^*UA = U \implies A$ is invertible and $A^{-1}$ is unitarily equivalent to $A^*$. Hence $\sigma(A) \subseteq \partial D$. Since $A$ is normal, $A$ is unitary. 

Remark 2.3. (i) Theorem 2.2(i) has a more satisfactory form for $pk - QH$ operators. Thus, if an operator $A \in pk - QH$ is such that $\delta_{A^*A}(X) = 0$ for some invertible operator $X$ such that $0 \notin \overline{W(X)}$, then $\sigma(A)$ is real. Hence $A_n$ in the decomposition $A = A_n \oplus A_p$ being normal is self-adjoint. Since a $pk - QH$ operator with zero Lebesgue area measure is the direct sum of a normal operator with a nilpotent operator [8, Corollary 6], $A_p$ is nilpotent and $A$ is the direct sum of a self-adjoint operator with a nilpotent operator. Hence: If $A \in pk - QH$, $0 \notin \overline{W(X)}$ and $\delta_{A^*A}(X) = 0$, then $A$ is the direct sum of a self-adjoint operator with a nilpotent operator (cf. [9, Theorem 5]). Observe that if the operator $A$ of Theorem 2.2(i) is reduction normaloid (i.e., the restriction of $A$ to reducing subspaces of $A$ is normaloid), then $A$ is self-adjoint. Although $pk - QH$ operators are not normaloid, $p - H$ operators are (reduction normaloid). Hence, if $A \in p - H$, $0 \notin \overline{W(X)}$ and $\delta_{A^*A}(X) = 0$, then $A$ is a self-adjoint operator [9, Theorem 2].

(ii) Let $A \in pk - QH$ and assume that $\triangle_{A^*A}(X) = 0$ for some invertible operator $X$. Then $A$ is left invertible. Hence, if $A$ has a dense range (evidently, see definition, such a $pk - QH$ operator is a $p - H$ operator), then $A$ is invertible, and so $\sigma(A) \subseteq \partial D$. Since a $p - H$ operator with spectrum in $\partial D$ is unitary, $A$ is unitary.

Applications. The restriction of a $pk - QH$ operator to an invariant subspace is again a $pk - QH$ operator. We assume in the following that $0 < p < 1$ and $k \geq 1$. Recall that every $A \in pk - QH \cap B(H)$ has a
representation
\[
\hat{A} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} T^k & \mathcal{H} \\ T^{*k-1} & (0) \end{pmatrix},
\]
where \(A_{11} \in p - H\) and \(A_{22}^k = 0\) [8]. Evidently \(\delta_{A_{11}N}^{-1}(0) \subseteq \delta_{A_{11}N^*}^{-1}(0)\) for every normal operator \(N\) and \(\delta_{A_{11}T}^{-1}(0) = \{0\}\) for all pure \(p\)-hyponormal or dominant operators \(T^*\) [2, Theorem 7 and Corollary 8]. If \(S\) is a hyponormal operator and \(T^*\) is a dominant operator, then \(\Delta_{ST}^{-1}(0) \subseteq \Delta_{S^*T^*}^{-1}(0)\): this is a consequence of the fact that \(\delta_{ST}^{-1}(0) \subseteq \delta_{S^*T^*}^{-1}(0)\) (see [1, Theorem 1] and [3, Theorem 2]). Since to every \(p\)-hyponormal operator \(R\) there corresponds a hyponormal operator \(S\) and a quasiaffinity \(X\) such that \(XR = RS\) [4, Proof of Theorem 1], it follows that \(\Delta_{A_{11}T}^{-1}(0) \subseteq \Delta_{A_{11}T^*}^{-1}(0)\) for every \(p\)-hyponormal or dominant operator \(T^*\).

**Theorem 2.4.** Let \(A \in pk - QH \cap B(\mathcal{H})\) and \(B \in B(K)\). If \(B^*\) is an either \(p\)-hyponormal or injective dominant or injective \(pk - QH\) operator (resp., \(B^*\) is an either \(p\)-hyponormal or dominant or \(pk - QH\) operator), then \(\delta_{AB}^{-1}(0) \subseteq \delta_{A^*B^*}^{-1}(0)\) (resp., \(\Delta_{A_{11}}^{-1}(0) \subseteq \Delta_{A_{11}^*T^*}^{-1}(0)\)).

**Proof.** Let \(d_{AB}\) stand for either of \(\delta_{AB}\) and \(\Delta_{AB}\). For a \(Y \in d_{AB}^{-1}(0)\), define the quasiaffinity \(X : \ker Y \to \overline{\operatorname{ran} Y}\) by setting \(Xx = Yx\) for each \(x \subseteq K\). Evidently, \(\overline{\operatorname{ran} Y}\) is invariant for \(A\) and \(\ker Y\) is invariant for \(B^*\), and \(d_{A,B_1}(X) = 0\), where \(A_1 = A|_{\overline{\operatorname{ran} Y}} \in pk - QH\) and \(B_1^* = B^*|_{\ker Y}\) is either \(p\)-hyponormal or injective dominant or an injective \(pk - QH\) operator. In view of our remarks above, it follows from Theorem 2.1 that \(B_1\) is normal and \(d_{A_1,B_1^*}(X) = 0\). Observe that \(d_{A_1,B_1}(X) = 0 = d_{A_1^*B_1^*}(X)\), \(B_1\) normal, implies that \(A_1\) is normal. It is not difficult to verify that invariant subspaces \(M\) of a \(pk - QH\) operator \(A\) such that \(A|M\) is injective normal are reducing [9, Lemma 10]; hence \(A = A_1 \oplus A_0\) for some operator \(A_0\). But then \(d_{A_1^*B_1^*}(X) = 0 \implies d_{A_1^*B_1^*}(X) = 0\). \(\square\)

**Remark 2.5.** (i) The hypothesis that \(B^*\) is an injective dominant or an injective \(pk - QH\) operator can not be relaxed in Theorem 2.4. Thus, let \(A = B = N \oplus K\), where \(N\) is normal and \(K\) is \(k\)-nilpotent. Then \(A\) and \(B^* \in pk - QH\). Let \(X_1\) be any operator in the commutant of \(N\), and let \(X = X_1 \oplus K^{k-1}\). Then \(\delta_{AB}(X) = 0\), but \(\delta_{A^*B^*}(X) \neq 0\). Again, let \(A = N \oplus K\), \(X = 0 \oplus K^{k-1}\) and \(B^* = D \oplus 0\) for some dominant operator \(D\). Then \(\delta_{AB}(X) = 0\), but \(\delta_{A^*B^*}(X) \neq 0\).

(ii) If \(A \in pk - QH\) is such that \(A|_{\overline{\operatorname{ran} A^k}}\) is normal, then \(\overline{\operatorname{ran} A^k}\) reduces \(A\). To see this, let \(A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} \overline{\operatorname{ran} A^k} \\ \ker A^* \end{pmatrix}\), where \(A_{11}\) is normal.
Let $A_{11} = N \oplus 0$, where $N$ is injective normal. Then [9, Lemma 10] implies that $A = \begin{pmatrix} N & 0 & 0 \\ 0 & 0 & A_2 \\ 0 & 0 & A_{22} \end{pmatrix} = N \oplus \begin{pmatrix} A_2 \\ 0 \\ A_{22} \end{pmatrix} = N \oplus A_0$. The operator $A_0$ is (of necessity) $pk$-quasihyponormal. It is easily verified that $A_0$ is $k+1$-nilpotent. Hence $\sigma(A_0) = \{0\}$. Applying [8, Corollary 6], it follows that $A_0$ is the direct sum of a normal with a nilpotent. Evidently, $A_2 = 0$.

Various combinations (such as $A \in pk - QH$ is injective and $B^* \in p - H$ (see [9, Theorem 11]) and variants (such as $A \in pk - QH$ and $B^* \in pk - QH$ such that $B^{*-1}(0)$ is reducing ($\implies$ the pure part $B^*_p$ of $B^*$ is injective)) are possible in Theorem 2.4: we leave the formulation of such combinations to the reader. A version of Theorem 2.4 holds for $pk - QH$ operators $A$ and spectral operators $B$. (See [5, Chapter XV] for information on spectral and scalar operators.)

**Theorem 2.6.** If $\delta_{AB}(X) = 0$ (resp., $\triangle_{AB}(X) = 0$) for some operator $A \in pk - QH$, spectral operator $B \in B(\mathcal{K})$ such that $\overline{B\mathcal{K}} = \overline{B^k\mathcal{K}}$ and quasiaffinity $X$ (resp., some operator $A \in pk - QH$ such that $0 \in \sigma(A) \implies 0 \in \sigma_p(A)$, spectral operator $B$ and quasiaffinity $X$), then $A$ is normal, $B$ is a scalar operator similar to $A$ and $\delta_{A^*B^*}(X) = 0$ (resp., $A$ is an invertible normal operator, $B$ is a scalar operator similar to $A$ and $\triangle_{A^*B^*}(X) = 0$).

**Proof.** Case $\delta_{AB}(X) = 0$. The hypothesis $\overline{B\mathcal{K}} = \overline{B^k\mathcal{K}}$ implies that $B \in B(\overline{B\mathcal{K}} \oplus B^{*-1}(0))$ has a representation $B = B_{11} \oplus 0$, where $B_{11}$ is spectral, and $X \in B(\overline{B\mathcal{K}} \oplus B^{*-1}(0), A^k\mathcal{H} \oplus A^{*k-1}(0))$ has a representation $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & 0 \end{pmatrix}$. Since $\text{ran}A^k\text{ran}X = \text{ran}X\overline{\text{ran}B^k} = \text{ran}X\overline{\text{ran}B}$, $X_{11}$ is a quasiaffinity. Hence $\delta_{A_{11}B_{11}}(X_{11}) = 0 \implies A_{11}$ is normal, $B_{11}$ is a scalar operator similar to $A_{11}$, and $\delta_{A_{11}^*B_{11}^*}(X_{11}) = 0$ [7, Theorem 11]. Since $X_{22}$ has dense range, $\delta_{AB}(X) = 0 \implies A_{22}X_{22} = 0 \implies A_{22} = 0$. Evidently, $A_{12} = 0$. (Observe that $A_{11}$ is normal $\implies$ $\text{ran}A^k$ reduces $A$; see Remark 2.5(ii).) Hence, $\delta_{A^*B^*}(X) = 0$, $A$ is normal and $B$ is a scalar operator similar to $A$.

Case $\triangle_{AB}(X) = 0$. Since $X$ is a quasiaffinity, and since $0 \not\in \sigma_p(A) \implies 0 \not\in \sigma(A)$, the hypothesis $\triangle_{AB}(X) = 0$ implies that $A$ is an invertible $p$-hyponormal operator such that $\delta_{A^{-1}B}(X) = 0$. Applying [7, Theorem 11], it follows that $A^{-1}$ is normal, $B$ is a scalar operator similar to $A^{-1}$ and $\delta_{A^{-1}B^*}(X) = 0 = \triangle_{A^*B^*}(X)$. \(\Box\)
Acknowledgements. Part of this work was done whilst the author was visiting ISI New-Delhi (India). The author thanks Prof. Rajendra Bhatia, and ISI, for their kind hospitality.

References


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