DIVERGENT CESÀRO MEANS OF FOURIER EXPANSIONS WITH RESPECT TO POLYNOMIALS ASSOCIATED WITH THE MEASURE 

$$(1 - x)\alpha (1 + x)^{\beta} + M\Delta_{-1}$$

Bujar Xh. Fejzullahu

Abstract

We prove that, for certain indices of $\delta$, there are functions whose Cesàro means of order $\delta$ of the Fourier expansion with respect to the polynomials associated with the measure $(1 - x)^\alpha (1 + x)^\beta + M\Delta_{-1}$, where $\Delta_t$ is the delta function at a point $t$, are divergent almost everywhere on $[-1, 1]$. We follow Meaney’s paper (2003), where divergent Cesàro and Riesz means of Jacobi expansions were proved.

1 Introduction

Let $d\mu$ be a finite positive Borel measure on the interval $I \subset R$ such that $\text{supp} (d\mu)$ is an infinite set and let $p_n(d\mu)$ denote the corresponding orthonormal polynomials. For $f \in L^1(I, d\mu)$, let $S_nf$ denote the $n$th partial sum of the orthonormal Fourier expansion of $f$ in $\{p_n(d\mu)\}_n=0^{\infty}$:

$$S_N f(x) = \sum_{n=0}^{N} c_n(f)p_n(x),$$

$$c_n(f) = \int_I f p_n d\mu.$$ 

The Cesàro means of order $\delta$ of the expansion (1) are defined by

$$\sigma^\delta_N f(x) = \sum_{n=0}^{N} A_N^{\delta-n} c_n(f)p_n(x),$$

2000 Mathematics Subject Classification. 42C05, 42C10.

Key words and phrases. Koornwinder’s Jacobi-type polynomials, Cesàro mean.

Received: May 22, 2007
where $A_k = \binom{k+\delta}{k}$.

The study of the convergence of Fourier series (1) with respect to polynomials associated with the measure $d\mu = (1-x)^\alpha(1+x)^\beta dx + M\Delta_{-1} + N\Delta_1$ has been discussed in [2] (see also [4]). If $\alpha, \beta \geq -1/2$, then (see [2])

$$||S_nf||_{L^p(d\mu)} \leq C ||f||_{L^p(d\mu)} \quad \forall n \geq 0, \; \forall f \in L^p(d\mu)$$

if and only if $p$ belongs to the open interval $(p_0, p_1)$, where

$$p_0 = \frac{4(\alpha + 1)}{2\alpha + 3}, \quad p_1 = \frac{4(\alpha + 1)}{2\alpha + 1}$$

when $\alpha \geq \beta$ and $\alpha > -1/2$ (and the analogous formulas with $\alpha$ replaced by $\beta$ if $\beta \geq \alpha$).

We show that, for $1 \leq p < p_0$ and $0 \leq \delta < 2\alpha + 3$, there are functions whose Fourier expansions associated to a measure $d\mu = (1-x)^\alpha(1+x)^\beta dx + M\Delta_{-1}$ have almost everywhere divergent Cesàro means of order $\delta$.

2 Koornwinder’s Jacobi-type polynomials

Let $\omega_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, $(\alpha, \beta > -1)$, be the Jacobi measure on the interval $[-1, 1]$. In [5] T. H. Koornwinder introduced the polynomials $\{P_n^{\alpha,\beta,M,N}(x)\}_{n=0}^\infty$ which are orthogonal on the interval $[-1, 1]$ with respect to the weight function

$$d\mu(x) = \frac{\Gamma(\alpha + \beta + 2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\omega_{\alpha,\beta}(x)dx + M\Delta_{-1}(x) + N\Delta_1(x),$$

where $\alpha > -1$, $\beta > -1$, $M \geq 0$, and $N \geq 0$. We call these polynomials the Koornwinder’s Jacobi-type polynomials.

We denote the orthonormal Koornwinder’s Jacobi-type polynomial by $p_n^{(\alpha,\beta,M,N)}$, which differs from $P_n^{(\alpha,\beta,M,N)}$ by normalization constant ([9, p. 81]).

Some basic properties of $p_n^{(\alpha,\beta,M,N)}$ (see [9, Chapter IV]), we will need in the following, are given below:

$$p_n^{(\alpha,\beta,M,N)}(1) \sim \begin{cases} n^{-\alpha-3/2} & \text{if } N > 0 \\ n^{\alpha+1/2} & \text{if } N = 0 \end{cases},$$

$$|p_n^{(\alpha,\beta,M,N)}(-1)| \sim \begin{cases} n^{-\beta-3/2} & \text{if } M > 0 \\ n^{\beta+1/2} & \text{if } M = 0 \end{cases}.$$
Divergent cesàro means of Fourier expansions with respect to \( u \) and \( v \) 

\[
|p_n^{(\alpha,\beta,M,N)}(\cos\theta)| = \begin{cases} 
O(\theta^{-\alpha-1/2}) & \text{if } c/n \leq \theta \leq \pi/2, \\
O(\theta^\alpha+1/2) & \text{if } 0 \leq \theta \leq c/n 
\end{cases}
\]  

(5)

for \( \alpha \geq -1/2, \beta \geq -1/2, \) and \( n \geq 1. \)

Asymptotic behaviour of the Jacobi orthonormal polynomials \( p_n^{(\alpha,\beta)} \), for \( x \in [-1+\epsilon, 1-\epsilon] \) and \( \epsilon > 0 \), it is given by (see [8, Theorem 8.21.8])

\[
p_n^{(\alpha,\beta)}(x) = r_n^{\alpha,\beta}(1-x)^{-\alpha/2-1/4} (1+x)^{-\beta/2-1/4} \cos(k\theta + \gamma) + O(n^{-1}),
\]

(6)

where \( x = \cos\theta, k = n + (\alpha + \beta + 1)/2, \gamma = -(\alpha + 1/2)\pi/2 \) and \( r_n^{\alpha,\beta} = 2^{(\alpha+\beta+1)/2}(\pi n)^{-1/2} \to (2/\pi)^{1/2}. \)

Now we will show that the polynomials \( p_n^{(\alpha,\beta,M,0)} \) have a similar asymptotic behaviour to the one of \( p_n^{(\alpha,\beta)}(x) \).

**Lemma 1.** Let \( p_n^{(\alpha,\beta,M,0)} \) be the polynomials orthonormal with respect to a measure \( d\mu = \omega_{\alpha,\beta} dx + M\Delta_{-1} \) and \( A_n, B_n \) the corresponding coefficients which appear in [3, Proposition 4]. Then, for \( x \in [-1+\epsilon, 1-\epsilon] \) and \( \epsilon > 0 \), we have

\[
p_n^{(\alpha,\beta,M,0)}(x) = s_n^{\alpha,\beta}(1-x)^{-\alpha/2-1/4} (1+x)^{-\beta/2-1/4} \cos(k\theta + \gamma) + O(n^{-1}),
\]

where \( s_n^{\alpha,\beta} = A_n r_n^{\alpha,\beta} + B_n r_n^{\alpha,\beta+2}. \)

**Proof.** By [3, Proposition 4]

\[
p_n^{(\alpha,\beta,M,0)}(x) = A_n p_n^{(\alpha,\beta)}(x) + B_n(x+1)p_n^{(\alpha,\beta+2)}(x).
\]

(7)

From (6), we have

\[
p_{n-1}^{(\alpha,\beta+2)}(x) = r_{n-1}^{\alpha,\beta+2}(1+x)^{-1}(1-x)^{-\alpha/2-1/4} (1+x)^{-\beta/2-1/4} \cos(k\theta + \gamma) + O(n^{-1}).
\]

Hence, from this and (6), we obtain

\[
p_n^{(\alpha,\beta,M,0)}(x) = (1-x)^{-\alpha/2-1/4} (1+x)^{-\beta/2-1/4} \cos(k\theta + \gamma) 
\]

\[
[A_n r_n^{\alpha,\beta} + B_n r_{n-1}^{\alpha,\beta+2}] + [A_n + B_n(x+1)]O(n^{-1})
\]

Since

\[
A_n \approx cn^{-2\beta-2}, \quad B_n \approx 1,
\]

(8)

see [8, p. 72, (4.5.8)] and [3, Proposition 4], where by \( u_n \approx v_n \) we mean that the sequence \( u_n/v_n \) converges to 1, we get

\[
p_n^{(\alpha,\beta,M,0)}(x) = s_n^{\alpha,\beta}(1-x)^{-\alpha/2-1/4} (1+x)^{-\beta/2-1/4} \cos(k\theta + \gamma) + O(n^{-1}).
\]

\[ \square \]
The formula of Mehler-Heine type for Jacobi orthonormal polynomials is (see [8, Theorem 8.1.1])

\[
\lim_{n \to \infty} n^{-\alpha-1/2} p_n^{(\alpha, \beta)}(\cos \frac{z}{n}) = c_{\alpha, \beta} (z/2)^{-\alpha} J_{\alpha}(z),
\]

where \( \alpha, \beta \) real numbers, \( c_{\alpha, \beta} \) is positive constant independent of \( n \) and \( z \), and \( J_{\alpha}(z) \) is the Bessel function. This formula holds uniformly for \(|z| \leq R\), \( R \) fixed.

Using (7), (8) and (9) we have:

**Lemma 2.** Let \( \alpha > -1, \beta > -1 \) and \( M > 0 \). Then

\[
\lim_{n \to \infty} n^{-\alpha-1/2} p_n^{(\alpha, \beta, M, 0)}(\cos \frac{z}{n}) = 2c_{\alpha, \beta+2} (z/2)^{-\alpha} J_{\alpha}(z),
\]

which holds uniformly for \(|z| \leq R\), \( R \) fixed.

For every function \( f \in L^1([-1, 1], d\mu) \) the Fourier coefficients of the series (1) can be written as

\[
c_n(f) = c'_n(f) + M f(-1) p_n^{(\alpha, \beta, M, N)}(-1) + N f(1) p_n^{(\alpha, \beta, M, N)}(1),
\]

where

\[
c'_n(f) = \frac{\Gamma(\alpha + \beta + 2)}{2^{\alpha+\beta+1} \Gamma(\alpha + 1) \Gamma(\beta + 1)} \int_{-1}^{1} f(x) p_n^{(\alpha, \beta, M, N)}(x) \omega_{\alpha, \beta}(x) dx.
\]

We next need to know the bounds for the integral involving Koorwinder’s Jacobi-type polynomials

\[
\int_{-1}^{1} |p_n^{(\alpha, \beta, M, N)}(x)|^q \omega_{\alpha, \beta}(x) dx
\]

where \( 1 \leq q < \infty \).

For \( M = N = 0 \) the calculation of this integral is in [8, p.391. Exercise 91] (see also [6]).

First we prove the upper bound for this integral:

**Theorem 1.** Let \( M \geq 0 \) and \( N \geq 0 \). For \( \alpha \geq -1/2 \)

\[
\int_{0}^{1} (1 - x)^{\alpha} |p_n^{(\alpha, \beta, M, N)}(x)|^q dx = \begin{cases} 
O(1) & \text{if } 2\alpha > q\alpha - 2 + q/2, \\
O(\log n) & \text{if } 2\alpha = q\alpha - 2 + q/2, \\
O(n^{q\alpha + q/2 - 2\alpha - 2}) & \text{if } 2\alpha < q\alpha - 2 + q/2.
\end{cases}
\]
Proof. From (5), for \( q\alpha + q/2 - 2\alpha - 2 \neq 0 \), we have

\[
\int_0^1 (1 - x)^\alpha |p_n^{(\alpha,\beta,M,N)}(x)|^q dx = O(1) \int_0^{\pi/2} \theta^{2\alpha + 1} |p_n^{(\alpha,\beta,M,N)}(\cos \theta)|^q d\theta
\]

\[
= O(1) \int_0^{\pi/2} \theta^{2\alpha + 1} n^{q\alpha + q/2} d\theta + O(1) \int_{n^{-1}}^{\pi/2} \theta^{2\alpha + 1} \theta^{-q\alpha - q/2} d\theta
\]

\[
= O(n^{q\alpha + q/2 - 2\alpha - 2}) + O(1),
\]

and for \( q\alpha + q/2 - 2\alpha - 2 = 0 \) we have

\[
\int_0^1 (1 - x)^\alpha |p_n^{(\alpha,\beta,M,N)}(x)|^q dx = O(\log n).
\]

Now using a technique similar to the one used in [8, Theorem 7.34] we obtain:

**Theorem 2.** Let \( M \geq 0 \) and \( N = 0 \). For \( \alpha \geq -1/2 \) and \( 2\alpha < q\alpha - 2 + q/2 \) we have

\[
\int_0^1 (1 - x)^\alpha |p_n^{(\alpha,\beta,M,0)}(x)|^q dx \sim n^{q\alpha + q/2 - 2\alpha - 2}
\]

**Proof.** For the proof of Theorem 2 it is sufficient to prove just the lower bound for the integral.

Let \( \alpha \geq -1/2 \) and \( M > 0 \). According to Lemma 2, we have

\[
\int_0^{\pi/2} \theta^{2\alpha + 1} |p_n^{(\alpha,\beta,M,0)}(\cos \theta)|^q d\theta > \int_0^{n^{-1}} \theta^{2\alpha + 1} |p_n^{(\alpha,\beta,M,0)}(\cos \theta)|^q d\theta
\]

\[
\cong c \int_0^1 (z/n)^{2\alpha + 1} n^{q\alpha + q/2} (z/2)^{-\alpha} J_\alpha(z) |^q n^{-1} dz \sim n^{q\alpha + q/2 - 2\alpha - 2}.
\]

\( \square \)
3 Divergent Cesàro means of the Fourier expansion with respect to polynomials associated with the measure 
\( (1 - x)^\alpha (1 + x)^\beta + M \Delta_{-1} \)

In [10, Theorem 3.1.22] it is proved

**Lemma 3.** Suppose that \( \lim_{N \to \infty} \sigma_N^\delta f(x) \) exists for some \( x \in [-1, 1] \) and \( \delta > -1 \). Then

\[
|c_N(f)p_N(x)| \leq C_\delta N^\delta \max_{0 \leq n \leq N} |\sigma_n^\delta f(x)|, \quad \forall N \geq 0.
\]

From Egorov’s theorem and Lemma 3 it follows that if the series (1) is Cesàro summable of order \( \delta \) on a set of positive measure in \([-1, 1]\) then there is a set of positive measure \( E \) on which

\[
|n^{-\delta} c_n(f)p_n^{(\alpha,\beta,M,0)}(x)| \leq A.
\]

Hence, from Lemma 1, we have

\[
|n^{-\delta} c_n(f) (\cos(k\theta + \gamma) + O(n^{-1}))| \leq A.
\]

uniformly for \( \cos \theta \in E \). Using the argument of the subsection 1.5 in [7] we obtain

\[
\left| \frac{c_n(f)}{n^\delta} \right| \leq A, \quad \forall n \geq 1. \quad (11)
\]

From Theorem 2, for \( \alpha > -1/2 \) and \( 1 \leq q < \infty \), we have

\[
\left( \int_{-1}^{1} |p_n^{(\alpha,\beta,M,0)}(x)|^q \omega_{\alpha,\beta}(x) dx \right)^{1/q} >
\]

\[
c \left( \int_{0}^{1} (1 - x)^\alpha |p_n^{(\alpha,\beta,M,0)}(x)|^q dx \right)^{1/q} \sim n^{\alpha+1/2-2\alpha/q-2/q} \quad (12)
\]

where \( q > \frac{4(\alpha+1)}{2\alpha+1} \).

For \( q = \infty \) and \( \alpha \geq \beta \geq -1/2 \) we have (see [9, (4.42), p.90])

\[
\max_{-1 \leq x \leq 1} |p_n^{(\alpha,\beta,M,0)}(x)| \sim p_n^{(\alpha,\beta,M,0)}(1) \sim n^{\alpha+1/2}. \quad (13)
\]

Now we are in position to prove our main result:
Theorem 3. Let given numbers $\alpha$, $\beta$, $p$, and $\delta$ be such that $\alpha > -1/2$;

\[-\frac{1}{2} \leq \beta \leq \alpha; \]
\[1 \leq p < \frac{4(\alpha + 1)}{2\alpha + 3}; \]
\[0 \leq \delta < \frac{2\alpha + 2}{p} - \frac{2\alpha + 3}{2}.\]

There is an $f \in L^p([-1, 1], \omega_{\alpha, \beta})$, supported in $[0, 1]$, whose Cesàro means $\sigma_N^\delta f(x)$ is divergent almost everywhere on $[-1, 1]$.

Proof. Suppose that
\[\delta < \frac{2\alpha + 2}{p} - \frac{2\alpha + 3}{2}.\]

For $q$ conjugate to $p$, from last inequality, we get
\[\delta < \alpha + \frac{1}{2} - \frac{2\alpha}{q} - \frac{2}{q}.\]

From the argument given in [7, Subsection 1.4], (12) and (13), for linear functional $c_n^\prime(f) = \frac{\Gamma(\alpha + \beta + 2)}{2^{\alpha + \beta + 1}\Gamma(\alpha + 1)\Gamma(\beta + 1)} \int_{-1}^{1} f(x)p_n^{(\alpha, \beta, M, 0)}(x)\omega_{\alpha, \beta}(x)dx$, it follows that there is an $f \in L^p([-1, 1], \omega_{\alpha, \beta})$, supported on $[0, 1]$, for which satisfy
\[\frac{c_n^\prime(f)}{n^\delta} \to \infty, \quad \text{as } n \to \infty.\]

Hence, from (10), we obtain
\[\frac{c_n(f)}{n^\delta} \to \infty, \quad \text{as } n \to \infty.\]

Since this result is contrary with (11) it follows that for this $f$ the $\sigma_N^\delta f(x)$ is divergent almost everywhere. \hfill \Box

Remark 1. Using formulae in [1], which relate the Riesz and Cesàro means of order $\delta \geq 0$, we conclude that Theorem 3 holds for Riesz means.

Remark 2. From the symmetry $P_n^{(\alpha, \beta, M, 0)}(-x) = (-1)^n P_n^{(\beta, \alpha, 0, M)}(x)$ we get the same results as above for the measure $d\mu = \omega_{\alpha, \beta}dx + N\Delta_1$. 
References


[2] J. J. Guadalupe, M. Pérez, F. J. Ruiz and J. L. Varona, *Convergence in the mean of the Fourier series with respect to polynomials associated with the measure $(1-x)^\alpha (1+x)^\beta + M\delta_1 + N\delta_1$*, Orthogonal polynomials and their applications (Spanish), (1989), 91-99.


Faculty of Mathematics and Sciences, University of Pristina
Mother Teresa 5, 10000 Pristina, Kosovo, Serbia

*E-mail: bujarfej@uni-pr.edu*