QUASI-NEARLY SUBHARMONIC FUNCTIONS AND CONFORMAL MAPPINGS

Vesna Kojić

Abstract

If $\varphi$ is a conformal mapping and $u$ is a quasi-nearly subharmonic function, then $u \circ \varphi$ is quasi-nearly subharmonic. A similar fact for “regularly oscillating” functions holds.

Introduction

If $u$ is a nonnegative subharmonic function on a domain $\Omega \subset \mathbb{C}$ and $p \geq 1$, then

$$u(z)^p \leq \frac{1}{r^2} \int_{B(z,r)} u^p \, dm$$

(1)

for all $B(x,r) \subset \Omega$. Here $B(z, r)$ is the Euclidean disk with center $z$ and radius $r$, and $dm$ is the Lebesgue measure in $\mathbb{C}$ normalized so that the measure of the unit ball equals one. If $0 < p < 1$, then (1) need not hold but there is a constant $C = C(p) \geq 1$ such that

$$u(z)^p \leq \frac{C}{r^2} \int_{B(z,r)} u^p \, dm$$

(2)

This fact, essentially due to Hardy and Littlewood [5], was first proved by Fefferman and Stein [3, Lemma 2, p. 172]. Fefferman and Stein’s proof is reproduced in Garnett [4, Lemma 3.7, pp. 121–123]. Although Fefferman

2000 Mathematics Subject Classification. Primary: 31B05, 31B25; Secondary: 31C05.

Key words and phrases. Subharmonic, quasi-nearly subharmonic, and regularly oscillating functions, Koebe’s theorems.

Received: August 20, 2007

The author is supported by MNZŽS Grant ON144010, Serbia
and Stein considered only the case when \( u = |v| \) and \( v \) is harmonic their proof applies also in the case of nonnegative subharmonic functions. Perhaps, the first simple proof of (2) was given in [10, p. 64], although it depended on the hypothesis that \( u \) was subharmonic. For other proofs see [13, p. 18, and Theorem 1, p. 19], (see also [14, Theorem A, p. 15]), [18, Lemma 2.1, p. 233], and [19, Theorem, p. 188]).

That the validity of (2) for some \( p \) implies its validity for all \( p \) was observed in [1, p. 132], [13, Theorem 1], and [18, Lemma 2.1]. See also [11], where the case of “\( M \)-subharmonic” functions was considered, and [8, 9], where some extensions of [11] were made.

For various applications of (2) we refer to [24, p. 191], [6, Theorems 1 and 2, pp. 117-118], [21, Theorems 1, 2 and 3, pp. 301, 307], [18, Theorem, p. 233], [22, Theorem 2, p. 271], [23, Theorem, p. 113], [15], [12], and [17]. Further information can be found in [19] and [17].

Quasi-nearly subharmonic functions

Let \( \Omega \) be a subdomain of the complex plane \( \mathbb{C} \). Following [20] and [17], we call a Borel measurable function \( u : \Omega \to [0, \infty] \) quasi-nearly subharmonic, if \( u \in L^1_{\text{loc}}(\Omega) \) and if there is a constant \( K = K(u, \Omega) \geq 1 \) such that

\[
u(z) \leq K \frac{1}{r^2} \int_{B(z,r)} u(w) \, dm(w) \tag{3}\]

for any disk \( B(z, r) \subset \Omega \).

In [18], the term pseudoharmonic functions is used, while in [13], condition (3) is called \( \text{sh}_K \)-condition. If \( K = 1 \) (and \( u \) takes its values in \( [-\infty, \infty] \)), then \( u \) is called nearly subharmonic (see [7]).

We will denote by \( QNS_K(\Omega) \) the class of all functions satisfying (3) (for a fixed \( K \)) and by \( QNS(\Omega) \) the class of all quasi-nearly subharmonic functions defined in \( \Omega \); so

\[
QNS(\Omega) = \bigcup_{K \geq 1} QNS_K(\Omega).
\]

One of the most important properties of \( QNS \) is the following fact, which generalizes the above mentioned result of Fefferman and Stein [3].

**Theorem A.** [1, 13, 18] If \( u \in QNS_K(\Omega) \), and \( p > 0 \), then \( u^p \in QNS_C(\Omega) \), where \( C \) is a constant depending only on \( p \), \( K \). In particular, if \( u^p \) is quasi-nearly subharmonic for some \( p > 0 \), then so is for every \( p > 0 \).
In this paper we prove the following:

**Theorem 1.** Let \( u \in QNS_K(\Omega) \) and \( \varphi \) a conformal mapping from a domain \( G \) onto \( \Omega \), then the composition \( u \circ \varphi \) belongs \( QNS_C(G) \), where \( C \) depends only on \( K \).

**Regularly oscillating functions**

A function \( f \) defined in \( \Omega \) is called regularly oscillating (see [17]) if \( f \) is of class \( C^1(\Omega) \) and

\[
|\nabla f(z)| \leq Kr^{-1} \sup_{B(z,r)} |f - f(z)|, \quad B(z,r) \subset \Omega. \tag{4}
\]

The class of such functions is denoted in [13] and [16] by \( OC^1_K(\Omega) \) (\( O = \) oscillation). The class of all regularly oscillating functions will be denoted by \( RO(\Omega) \).

**Theorem B.** [13, Theorem 3] If \( f \) is regularly oscillating, then \(|f|\) and \(|\nabla f|\) are quasi-nearly subharmonic. Moreover, if \( f \in OC^1_K(\Omega) \), then \(|f|\) and \(|\nabla f|\) are in \( QNS_C(\Omega) \), where \( C \) depends only on \( K \).

**Example 1.** Harmonic functions are regularly oscillating.

**Example 2.** [13] Convex functions are regularly oscillating. It follows that the modulus of the gradient of a convex function is quasi-nearly subharmonic.

**Example 3.** [14] If \( f \) is an eigenfunction of \( \Delta \), i.e., \( \Delta f = \lambda f \) for some constant \( \lambda \), and if \( \Omega \) is bounded, then \( f \in OC^2(\Omega) \).

**Example 4 (Polyharmonic functions).** A function \( f \in C^{2k}(\Omega) \) is said to be polyharmonic (of degree \( k \)) if it is annihilated by the \( k \)-th power of the Laplacian. It is proved in [14, Corollary 5] (see also [15]) that every polyharmonic function is regularly oscillating, and therefore \(|f|\) and \(|\nabla f|\) are quasi-nearly subharmonic.

Here we prove the following:

**Theorem 2.** If \( f \in RO(\Omega) \), and \( \varphi \) is a conformal mapping from \( G \) onto \( \Omega \), then \( f \circ \varphi \in G \). Moreover if \( f \in OC^1_K(\Omega) \), then \( f \circ \varphi \) is in \( OC^1_C(G) \), where \( C \) depends only on \( K \).
Proofs

Our proofs are based on two theorems of Koebe (see [2, Theorem 2.3 and Theorem 2.5].

Theorem C (Koebe one-quarter theorem). Let \( \varphi \) be a conformal mapping from the disk \( B(z_0, R) \) into \( \mathbb{C} \), then the image \( \varphi(B(z_0, R)) \) contains the disk \( B(\varphi(z_0), \rho) \), where \( \rho = R|\varphi'(z_0)|/4 \).

Theorem D (Koebe distortion theorem). Let \( \varphi \) be a conformal mapping from the disk \( B(z_0, R) \) into \( \mathbb{C} \), then there holds the inequalities

\[
\frac{R^2(R - |z - z_0|)}{(R + |z - z_0|)^3} \leq \frac{R^2(R + |z - z_0|)}{(R - |z - z_0|)^3}, \quad z \in B(z_0, R).
\]

Consequently if \( |z - z_0| < R/2 \), then

\[
\frac{|\varphi'(z)|}{|\varphi'(z_0)|} \geq \frac{4}{27}.
\]

Proof of Theorem 1.

Let \( u \in QNS_K(\Omega) \) and \( \varphi \) a conformal mapping from \( G \) onto \( \Omega \). We have to find a constant \( C \) such that

\[
\int_{B(z_0,r)} u(\varphi(z)) \, dm(z) \geq u(\varphi(z_0))r^2/C,
\]

whenever \( r < \text{dist}(z, \partial G) \). Let \( w_0 = \varphi(z_0) \) and \( \rho = r|\varphi'(z_0)|/4 \), and let \( \psi : \Omega \mapsto G \) denote the inverse of \( \varphi \). Then

\[
\int_{B(z_0,r)} u(\varphi(z)) \, dm(z) = \int_{\varphi(B(z_0,r))} u(w)|\psi'(w)|^2 \, dm(w)
\geq \int_{B(w_0,\rho/2)} u(w)|\psi'(w)|^2 \, dm(w),
\]

where we have applied the one-quarter theorem. Now we apply the distortion theorem to the function \( \psi \) to get \( |\psi'(w)| \geq (4/27)|\psi'(w_0)| \), for \( |w - w_0| < \rho/2 \). It follows that

\[
\int_{B(z_0,r)} u(\varphi(z)) \, dm(z) \geq (4/27)^2|\psi'(w_0)|^2 \int_{B(w_0,\rho/2)} u(w) \, dm(w)
\geq (4/27)^2|\psi'(w_0)|^2(\rho/2)^2u(w_0)/K
= (4/27)^2|\psi'(w_0)|^2|\varphi'(z_0)|^2u(w_0)r^2/16K.
\]
Now we use the identity \( \psi'(w_0)\varphi'(z_0) = 1 \) to get (†) with \( C = 27^2K \). This concludes the proof of Theorem 1.

**Proof of Theorem 2.**

Let \( u \in OC^1_K(\Omega) \) and \( \varphi \) a conformal mapping from \( G \) onto \( \Omega \). We have to find a constant \( C_1 \) such that

\[
|\nabla u(\varphi(z_0))| \cdot |\varphi'(z_0)| \leq \frac{C_1}{\varepsilon} \sup_{z \in B(z_0, \varepsilon)} |u(\varphi(z)) - u(\varphi(z_0))|, \quad B(z_0, \varepsilon) \subset G.
\]  

Let \( w_0 = \varphi(z_0) \) and \( \rho = \varepsilon |\varphi'(z_0)|/4 \), and let \( \psi \) be the inverse of \( \varphi \). Then, the definition of \( OC^1_K \) and by the one-quarter theorem,

\[
sup\{|u(\varphi(z)) - u(\varphi(z_0))| : z \in B(z_0, \varepsilon)\} \\
= sup\{|u(w) - u(w_0)| : w \in \varphi(B(z_0, \varepsilon))\} \\
\geq sup\{|u(w) - u(w_0)| : w \in B(w_0, \rho)\} \\
\geq |\nabla u(w_0)|\rho/K \\
= |\nabla u(w_0)| \cdot |\varphi'(z_0)|\varepsilon/4K.
\]

This gives (7) with \( C_1 = 4K \), concluding the proof.

**References**


Fakultet organizacionih nauka, Jove Ilića 154, Belgrade, Serbia

E-mail: vesnak@fon.bg.ac.yu