A COHEN TYPE INEQUALITY FOR LEGENDRE-SOBOLEV EXPANSIONS

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Abstract
Let introduce the Sobolev-type inner product
\[ \langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx + N[f'(1)g'(1) + f'(-1)g'(-1)], \]
where \( N \geq 0 \). In this paper we prove a Cohen type inequality for Fourier expansion in terms of the polynomials associated to the Sobolev inner product.

1 Introduction and Main Result
The purpose of this paper is to derive a lower bound for the norm associated to the Sobolev spaces of the polynomial expansions relative to Sobolev inner product. For classical orthogonal expansions such inequalities were proved by Dreseler and Soardi [5] and Markett [7].

Let us first introduce some notation. We shall say that \( f \in L^p \) if \( f \) is measurable on the \([-1,1]\) and \( \|f\|_{L^p} < \infty \), where
\[
\|f\|_{L^p} = \begin{cases} \left( \int_{-1}^{1} |f(x)|^p dx \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \text{ess sup} |f(x)| & \text{if } p = \infty. \end{cases}
\]

Now let us introduce the Sobolev spaces
\[
S_p = \{ f : \|f\|_{S_p}^p = \|f\|_{L^p}^p + N[|f'(1)|^p + |f'(-1)|^p] < \infty \}, \quad 1 \leq p < \infty,
\]
\[
S_\infty = \{ f : \|f\|_{S_\infty} = \|f\|_{L^\infty(d\mu)} < \infty \}, \quad p = \infty.
\]

2000 Mathematics Subject Classifications. 42C05, 42C10.
Key words and Phrases. Legendre-Sobolev polynomials, polynomial expansions, Cohen type inequality.
Received: August 21, 2007
Communicated by Dragan S. Djordjević
We also introduce the discrete Sobolev-type inner product
\[
\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx + N[f'(1)g'(1) + f'(-1)g'(-1)]
\]
for any functions \( f, g \) for which the right side makes sense and \( N \geq 0 \). We denote by \( \hat{B}_n \) the orthonormal polynomials with respect to the inner product (1) (see [2], [3], [6]). We call these polynomials the Legendre-Sobolev polynomials.

For \( N = 0 \), denoted by \( p_n \), we have classical Legendre orthonormal polynomials.

For \( f \in S_1 \), the Fourier expansion in Legendre-Sobolev polynomials is
\[
\sum_{k=0}^{\infty} \hat{f}(k)\hat{B}_k(x),
\]
\( \hat{f}(k) = \langle f, \hat{B}_k \rangle \).

The Cesàro means of order \( \delta \) of the expansion (2) are defined by (see [9, p. 76-77])
\[
\sigma_\delta^n f(x) = \sum_{k=0}^{n} \frac{A_{\delta}^{n-k}}{A_n} \hat{f}(k)\hat{B}_k(x),
\]
where \( A_{\delta}^k = \binom{k+\delta}{k} \).

For a function \( f \in S_p \) and a given sequence \( \{c_{k,n}\}_{k=0}^{n} \) of complex numbers with \( |c_{n,n}| > 0 \), we define the operator \( T_n^N \) by
\[
T_n^N (f) = \sum_{k=0}^{n} c_{k,n} \hat{f}(k)\hat{B}_k.
\]

Now we formulate main result

**Theorem 1.** Let \( 1 \leq p \leq \infty \). There exists a positive constant \( c \), independent of \( n \), such that
\[
\|T_n^N\|_{[S_p]} \geq c|c_{n,n}|egin{cases}
(n^{p-1})^{\frac{1}{2}} & \text{if } 1 \leq p < 4/3 \\
(\log n)^{\frac{1}{2}} & \text{if } p = 4/3, \ p = 4 \\
n^{1-p} & \text{if } 4 < p \leq \infty,
\end{cases}
\]
where by \([S_p]\) we denote the space of all bounded, linear operators from a space \( S_p \) into itself, furnished with the usual operator norm \( \| \cdot \|_{[S_p]} \).

**Corollary 1.** Let \( 1 \leq p \leq \infty \). For \( c_{k,n} = 1, k = 0, \ldots, n \), and for \( p \) outside the Pollard interval \((4/3, 4)\)
\[
\|S_n\|_{[S_p]} \to \infty, \quad n \to \infty,
\]
where \( S_n \) denotes the \( n \)th partial sum of expansion (2).
For $c_{k,n} = \frac{A^\delta_{n-k}}{A^\delta_n}$, $0 \leq k \leq n$, the Theorem 1 yield:

**Corollary 2.** Let given numbers $p$ and $\delta$ such that $1 \leq p \leq \infty$;

\[
\begin{align*}
0 < \delta < \frac{2}{p} - \frac{3}{2} & \quad \text{if } 1 \leq p < 4/3 \\
0 < \delta < \frac{1}{2} - \frac{2}{p} & \quad \text{if } 4 < p \leq \infty.
\end{align*}
\]
Then, for $p \notin [4/3, 4]$,

\[\|\sigma^\delta_n\|_{[S^p]} \to \infty, \quad n \to \infty.\]

## 2 Preliminaries

We summarize the properties of Legendre-Sobolev polynomials we need, cf. [6] (see also [2], [3]).

Let $\mu$ be the Gegenbauer (or ultraspherical) measure, $d\mu(x) = (1 - x^2)^\alpha dx$, $\alpha > -1$, and let $p^{(\alpha)}_n$ the corresponding Gegenbauer orthonormal polynomials.

The representation of $\hat{B}_n$ is

\[
\hat{B}_n(x) = A_n(1 - x^2)^{p^{(4)}_n} + B_n(1 - x^2)p^{(2)}_{n-2}(x) + C_np_n(x)
\]

where

\[
A_n \cong 1, \quad B_n \cong -2, \quad C_n \cong -1
\]

and by $u_n \cong v_n$ we mean that the sequence $u_n/v_n$ converges to 1.

The maximum of $\hat{B}_n$ on $[-1, 1]$ is

\[
\max_{x \in [-1, 1]} |\hat{B}_n(x)| \sim n^{1/2}
\]

where by $u_n \sim v_n$ we mean that there exist some positive constants $c_1$ and $c_2$ such that $c_1u_n \leq v_n \leq c_2u_n$ for sufficiently large $n$.

The polynomials $\hat{B}_n$ satisfy the estimate

\[
|\hat{B}_n(\cos \theta)| = \begin{cases} O(\theta^{-1/2}) & \text{if } c/n \leq \theta < \pi/2 \\ O(n^{1/2}) & \text{if } 0 \leq \theta \leq c/n \end{cases}
\]

and $c$ is positive constant.

The formula of Mehler-Heine type for Gegenbauer orthonormal polynomials is (see [8, Theorem 8.1.1] and [8, (4.3.4)])

\[
\lim_{n \to \infty} n^{-\alpha-1/2}p^{(\alpha)}_n(\cos \frac{z}{n}) = z^{-\alpha}J_\alpha(z),
\]

where $\alpha$ real number and $J_\alpha(z)$ is the Bessel function. This formula holds uniformly for $|z| \leq R$, $R$ fixed.

From (6) it can be shown that

\[
\lim_{n \to \infty} n^{-\alpha-1/2}p^{(\alpha)}_n(\cos \frac{z}{n+j}) = z^{-\alpha}J_\alpha(z)
\]

holds uniformly for $|z| \leq R$, $R$ fixed, and $j \in N \cup \{0\}$. 
Lemma 1. Let $N > 0$. Then
\[ \lim_{n \to \infty} n^{-1/2} \hat{B}_n(\cos \frac{z}{n}) = J_4(z) - 2J_2(z) - J_0(z) \]
which holds uniformly for $|z| \leq R$, $R$ fixed.

Proof. From (3) we have
\[ n^{-1/2} \hat{B}_n(\cos \frac{z}{n}) = A_n \sin^4 \left( \frac{z}{n} \right) n^{-1/2} p_{n-4}(\cos \frac{z}{n}) \]
\[ + B_n \sin^2 \left( \frac{z}{n} \right) n^{-1/2} p_{n-2}(\cos \frac{z}{n}) + C_n n^{-1/2} p_n(\cos \frac{z}{n}). \]

Finally, we take the limit $n \to \infty$ and use (3) and (7) to obtain
\[ \lim_{n \to \infty} n^{-1/2} \hat{B}_n(\cos \frac{z}{n}) = z^4z^{-3}J_4(z) - 2z^2z^{-2}J_2(z) - J_0(z) = J_4(z) - 2J_2(z) - J_0(z). \]

We also need to know the $S_p$ norms for Jacobi-Sobolev polynomials
\[ \| \hat{B}_n \|_{S_p}^p = \int_{-1}^{1} |\hat{B}_n(x)|^p dx + N|\hat{B}_n(1)|^p + N|\hat{B}_n(-1)|^p \]
where $1 \leq p < \infty$. Hence, it is sufficient to estimate just the $L^p$ norms for $\hat{B}_n$. For $N = 0$ the calculation of these norms is in [8, p.391, Exercise 91] (see also [7, (2.2)]).

Lemma 2. Let $N \geq 0$. Then
\[ \int_{0}^{1} |\hat{B}_n(x)|^p dx \sim \begin{cases} c & \text{if } p < 4, \\ \log n & \text{if } p = 4, \\ n^{p/2-2} & \text{if } p > 4. \end{cases} \]

Proof. From (5), for $p \neq 4$, we have
\[ \int_{0}^{1} |\hat{B}_n(x)|^p dx \sim \int_{0}^{\pi/2} \theta |\hat{B}_n(\cos \theta)|^p d\theta \]
\[ = O(1) \int_{0}^{n^{-1}} \theta n^{p/2} d\theta + O(1) \int_{n^{-1}}^{\pi/2} \theta^{-p/2} d\theta = O(n^{p/2-2}) + O(1), \]
and for $p = 4$ we have
\[ \int_{0}^{1} |\hat{B}_n(x)|^4 dx = O(\log n). \]
On the other hand, according to Lemma 1, we have

\[
\int_0^{\pi/2} \theta |\hat{B}_n(\cos \theta)|^p d\theta > \int_0^{n-1} \theta |\hat{B}_n(\cos \theta)|^p d\theta \approx c \int_0^1 \frac{(z/n)^{p/2}}{n^{-1}} |J_4(z) - 2J_2(z) - J_0(z)|^p dz \sim n^{p/2-2}.
\]

Using a similar argument as above, for \( p = 4 \), we have

\[
\int_0^{\pi/2} \theta |\hat{B}_n(\cos \theta)|^4 d\theta > c \int_0^{n-1} \theta |\hat{B}_n(\cos \theta)|^4 d\theta \approx c \int_0^1 \frac{z}{n^{1/2}} |J_4(z) - 2J_2(z) - J_0(z)|^4 dz \sim n \geq c \log n.
\]

Finally, from (3) and [8, Theorem 8.21.8] we obtain

\[
\int_0^{\pi/2} \theta |\hat{B}_n(\cos \theta)|^p d\theta > \int_0^{\pi/2} \theta |\hat{B}_n(\cos \theta)|^p d\theta \sim c.
\]

\[\square\]

3 Proof of Theorem 1

For the proof of Theorem 1 we will use the test function

\[g_n^j(x) = (1 - x^2)^j p_n^{(j)}(x)\]

where \( j \in \mathbb{N} \setminus \{1\} \). From (2) and (3) the Fourier coefficients of the function \( g_n^j(x) \) can be written as

\[(g_n^j)^\prime(k) = \int_{-1}^{1} (1 - x^2)^j p_n^{(j)}(x) \hat{B}_k(x) dx = A_k \int_{-1}^{1} (1 - x^2)^j p_n^{(j)}(x) (1 - x^2)^4 p_{n-4}^{(4)}(x) dx + B_k \int_{-1}^{1} (1 - x^2)^j p_n^{(j)}(x) (1 - x^2)^2 p_{n-2}^{(2)}(x) dx + C_k \int_{-1}^{1} (1 - x^2)^j p_n^{(j)}(x) p_k(x) dx = I_1^{k,n} + I_2^{k,n} + I_3^{k,n}\]

where it is assumed \( p_i^{(\alpha)}(x) = 0 \), for \( i = -1, -2, -3, -4 \).

For \( k \geq 4 \), according to the [8, (4.3.4)] we obtain

\[I_1^{k,n} = A_k \{h_n^j l^{-1/2} \{h_{k-4}^{j,4} \}^{-1/2} \int_{-1}^{1} (1 - x^2)^j P_n^{(j)}(x) (1 - x^2)^2 P_{k-4}^{(4)}(x) dx,\]
where \( h_n^{\alpha} = 2^{2\alpha} n^{-1} \).

On the other hand, from [7, (2.8)]

\[
(1 - x^2)^j P_n^{(j)}(x) = \sum_{m=0}^{2j} b_{m,j}(0,0,n) P_{n+m}(x)
\]

and

\[
(1 - x^2)^2 P_{k-4}^{(4)}(x) = \sum_{l=0}^{4} b_{l,2}(2,2,k - 4) P_{k+l-4}^{(2)}(x),
\]

where

\[
b_{0,j}(\alpha, \alpha, n) = 4^j \frac{(\Gamma(n + \alpha + j + 1)^2}{(\Gamma(n + \alpha + 1)^2} \frac{\Gamma(2n + 2\alpha + 2)}{\Gamma(2n + 2\alpha + 2j + 2)} \approx 4^j,
\]

\[
b_{2j,j}(\alpha, \alpha, n) = (-4)^j \frac{\Gamma(n + 2j + 1)}{\Gamma(n + 1)} \frac{\Gamma(2n + 2\alpha + 2j + 1)}{\Gamma(2n + 2\alpha + 4j + 1)} \approx (-4)^j.
\]

Thus

\[
\begin{cases}
I_{k,n}^{j} = 0, & 4 \leq k \leq n - 1 \\
I_{n,n}^{j} = 0, & n \geq 4, 0 < m \leq 2j.
\end{cases}
\]

Let \( k = n \geq 4 \) and \( m = 0 \). Then

\[
I_{n,n}^{j} = A_n \left\{ h_{n}^{j,j} \right\}^{-1/2} \left\{ h_{n-4}^{4,4} \right\}^{-1/2} b_{0,j}(0,0,n) b_{4,2}(2,2,n - 4) \int_{-1}^{1} P_{n}(x) P_{n}^{(2)}(x) dx
\]

Since (see [1, p. 359, Theorem 7.1.4])

\[
P_{n}^{(2)}(x) = \frac{16(n + 1/2)(n + 3/2)}{(n + 3)(n + 4)} P_{n} + Q_{n-1}(x),
\]

we get

\[
I_{n,n}^{j} = \frac{16A_n(n + 1/2)(n + 3/2)}{(n + 3)(n + 4)} \left\{ h_{n}^{j,j} \right\}^{-1/2} \left\{ h_{n-4}^{4,4} \right\}^{-1/2} h_{0,0}^{j,j} \times b_{0,j}(0,0,n) b_{4,2}(2,2,n - 4) \approx 16 \cdot 2^j.
\]

In similar way, for \( k \geq 2 \), using [8, (4.3.4)] and (9)

\[
I_{k,n}^{j} = B_k \left\{ h_{n}^{j,j} \right\}^{-1/2} \left\{ h_{k-2}^{2,2} \right\}^{-1/2} \sum_{m=0}^{2j} b_{m,j}(0,0,n) \int_{-1}^{1} P_{n+m}(x) (1 - x^2) P_{k}^{(2)}(x) dx.
\]

Again, as applications of [7, (2.8)] and [1, p. 359, Theorem 7.1.4] we point out the following relations

\[
(1 - x^2) P_{k-2}^{(2)}(x) = \sum_{l=0}^{2} b_{l,1}(1,1,k - 2) P_{k+l-2}^{(1)}(x).
\]
and
\[ P_n^{(1)}(x) = \frac{4n + 2}{n + 2} P_n + Q_{n-1}(x). \]

Thus
\[
\begin{cases}
  I_{k,n}^2 = 0, & 2 \leq k \leq n - 1 \\
  I_{n,n}^2 = \frac{(4n+2)b_{n+2}}{n+2} \left\{ h_{n-j}^{1/2} h_{n-2}^{1/2} - 1/2 \right\} h_n^{0,0} \\
  \quad \quad \quad \quad \quad \quad \quad \quad \quad \times b_{0,j}(0,0,n) b_{2,1}(1,1,n-2) \cong 8 \cdot 2^j & n \geq 2, \ m = 0 \\
  I_{n,n}^2 = 0 & n \geq 2, \ 0 < m \leq 2j.
\end{cases}
\]

Finally, for \( k \geq 0 \)
\[
I_{k,n}^3 = C_k \left\{ h_{n-j}^{1/2} \right\} h_{n}^{0,0} \left\{ h_{k}^{0,0} \right\} b_{m,j}(0,0,n) \int_{-1}^{1} P_{n+m}(x) P_k(x) dx.
\]

Thus
\[
\begin{cases}
  I_{k,n}^3 = 0, & 0 \leq k \leq n - 1 \\
  I_{n,n}^3 = C_n \left\{ h_{n-j}^{1/2} \right\} h_{n}^{0,0} \left\{ h_{k}^{0,0} \right\} b_{0,j}(0,0,n) \cong -2^j & n \geq 0, \ m = 0 \\
  I_{n,n}^3 = 0, & n \geq 0, \ 0 < m \leq 2j.
\end{cases}
\]

As a conclusion
\[
\begin{cases}
  (g_{j,n}^i) \cdot (k) = 0, & 0 \leq k \leq n - 1 \\
  (g_{j,n}^i) \cdot (n) \cong -2^j, & n = 0, 1 \\
  (g_{j,n}^i) \cdot (n) \cong 6 \cdot 2^j, & n = 2, 3 \\
  (g_{j,n}^i) \cdot (n) \cong 23 \cdot 2^j, & n \geq 4.
\end{cases} \tag{10}
\]

On the other hand, from [8, p.391. Exercise 91] (see also [7, (2.2)])
\[
\| g_n^j \|_p = \| g_n^j \|_{L_p} = \int_{-1}^{1} (1 - x)^{jp} |(1 + x)^{jp} p_n^{(j)}(x)|^p dx \\
\leq c_1 \int_{0}^{1} (1 - x)^{jp} |p_n^{(j)}(x)|^p dx \\
+ c_2 \int_{-1}^{0} (1 + x)^{jp} |p_n^{(j)}(x)|^p dx \leq c. \tag{11}
\]

for \( j > 1/2 - 2/p \) and \( 4 \leq p < \infty \).

It is well known (see, for example, [4, Theorem 1]) that
\[
|p_n^{(j)}(x)| \leq c(1 - x^2)^{-j/2-1/4}
\]

for \( x \in (-1, 1) \).

Therefore
\[
\| g_n^j \|_{S_m} = \| g_n^j \|_{L_m} \leq c(1 - x^2)^{j/2-1/4} \leq c, \tag{12}
\]
for $j > 1/2$.

Now we are in position to prove our main result:

**Proof of Theorem 1.** By duality, it suffices to assume that $4 \leq p \leq \infty$. Now we apply the operator $T^N_n$ to the test function $g^l_j$ for some $j > 1/2 - 2/p$.

Hence, from (10), (11) and (12), we have

$$
\| T^N_n \|_{S_p} \geq \| g^l_n \|_{S_p}^{-1} \| T^N_n g^l_j \|_{S_p} \geq c|c_{n,n}| \| \hat{B}_n \|_{S_p}. 
$$

(13)

From (8) and Lemma 2 we obtain that

$$
\| \hat{B}_n \|_{S_p} \geq c \begin{cases} (\log n)^{1/p} & \text{if } p = 4, \\ n^{1/2-2/p} & \text{if } 4 < p < \infty. \end{cases}
$$

On combining this and (4) with (13), the statement is seen to be true.

**References**


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