ON GO-COMPACT SPACES

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Abstract
The purpose of this paper is to offer some more properties of GO-compact spaces and to introduce and investigate some properties of \( g \)-continuous multifunctions. We also investigate GO-compact spaces in the context of multifunctions.

1 Introduction and preliminaries
Levine [11] introduced the concept of generalized closed sets of a topological space. Since the advent of these notions, several research papers with interesting results in different respects came to existence (see, [1], [3], [4], [5], [6], [8], [9], [10], [12]). Recently Caldas and Jafari [4] introduced and investigated the concepts of \( g \)-US spaces, \( g \)-convergency, sequential GO-compactness, sequential \( g \)-continuity and sequential \( g \)-sub-continuity.

Throughout the present paper \((X, \tau)\) and \((Y, \sigma)\) (or simply \(X\) and \(Y\)) denote topological spaces. Let \(A\) be a subset of \(X\). We denote the interior and the closure of \(A\) by \(\text{Int}(A)\) and \(\text{Cl}(A)\), respectively. \(A \subset X\) is called a generalized closed set (briefly \(g\)-closed set) of \(X\) [11] if \(\text{Cl}(A) \subset G\) holds whenever \(A \subset G\) and \(G\) is open in \(X\). The union of two \(g\)-closed sets is a \(g\)-closed set. A subset \(A\) of \(X\) is called a \(g\)-open set of \(X\), if its complement \(A^c\) is \(g\)-closed in \(X\). The intersection of all \(g\)-closed sets containing a set \(A\) is called the \(g\)-closure of \(A\) [10] and is denoted by \(\text{gCl}(A)\). If \(A \subset X\), then \(A \subset \text{gCl}(A) \subset \text{Cl}(A)\). The collection of all \(g\)-closed (resp. \(g\)-open) subsets of \(X\) will be denoted by \(\text{GC}(X)\) (resp. \(\text{GO}(X)\)). We set \(\text{GC}(X, x) = \{V \in \text{GC}(X) / x \in V\}\) for \(x \in X\). We define similarly \(\text{GO}(X, x)\). Let \(p\) be a point of \(X\) and \(N\) be a subset of \(X\). \(N\) is called a \(g\)-neighborhood of \(p\) in \(X\) [3] if there exists a \(g\)-open set \(O\) of \(X\) such that \(p \in O \subset N\).

A space \(X\) is \(GO\)-compact if every \(g\)-open cover of \(X\) has a finite subcover. Since every open sets is a \(g\)-open set, it follows that every \(GO\)-compact space is compact.
But, the converse may be false. Let \( X = \{ x \} \cup \{ x_i : i \in I \} \) where the indexed set
\( I \) is uncountable. Let \( \tau = \{ \emptyset, \{ x \}, X \} \) be the topology on \( X \). Evidently, \( X \) is a
compact space. However, it is not a \( GO \)-compact space because \( \{ \{ x, x_i \} : i \in I \} \) is
a \( g \)-open covering of \( X \) but it has no finite subcover. A subset \( A \) of a space \( X \) is said
to be \( GO \)-compact if \( A \) is \( GO \)-compact as a subspace of \( X \). If the product space
of two no-empty spaces is \( GO \)-compact, then each factor space is \( GO \)-compact \([1]\).
If \( A \) is \( g \)-open in \( X \) and \( B \) is \( g \)-open in \( Y \), then \( A \times B \) is \( g \)-open in \( X \times Y \) \([11]\).
A function \( f : X \rightarrow Y \) is said to be \( g \)-continuous \([1]\) if the inverse image of every
closed set in \( Y \) is \( g \)-closed in \( X \).

It is the purpose of this paper to offer some more characterizations of \( GO \)-
compact spaces. We also introduce the notion of \( g \)-complete accumulation points
by which we give some characterizations of \( GO \)-compact spaces. By introducing
the notion of 1-lower (resp. 1-upper) \( g \)-continuous functions and considering the
known notion of 1-lower (resp. 1-upper) compatible partial orders we investigate
some more properties of \( GO \)-compactness. We also investigate \( GO \)-compact spaces
in the context of multifunctions by introducing 1-lower(resp. 1-upper) \( g \)-continuous
multifunctions. Lastly we also obtain some characterizations of \( GO \)-compact spaces
by using lower (resp. upper) \( g \)-continuous multifunctions. In this paper we are
working in ZFC.

Recall that a function \( f : X \rightarrow Y \) is said to be \( g \)-continuous \([3]\) if the inverse
image of each open set in \( Y \) is \( g \)-open in \( X \).

Let \( \Lambda \) be a directed set. Now we introduce the following notions which will be
used in this paper. A net \( \xi = \{ x_\alpha \mid \alpha \in \Lambda \} \) \( g \)-accumulates at a point \( x \in X \) if
the net is frequently in every \( U \in GO(X, x) \), i.e. for each \( U \in GO(X, x) \) and for
each \( \alpha_0 \in \Lambda \), there is some \( \alpha \geq \alpha_0 \) such that \( x_\alpha \in U \). The net \( \xi \)
\( g \)-converges to a point \( x \) of \( X \) if it is eventually in every \( U \in go(X, x) \). We say that a filterbase
\( \Theta = \{ F_\alpha \mid \alpha \in I \} \) \( g \)-accumulates at a point \( x \in X \) if \( x \in \bigcap_{\alpha \in I} gCl(F_\alpha) \). Given a
set \( S \) with \( S \subset X \), a \( g \)-cover of \( S \) is a family of \( g \)-open subsets \( U_\alpha \) of \( X \)
for each \( \alpha \in I \) such that \( S \subset \bigcup_{\alpha \in I} U_\alpha \). A filterbase \( \Theta = \{ F_\alpha \mid \alpha \in I \} \) \( g \)-converges to a point
\( x \) in \( X \) if for each \( U \in GO(X, x) \), there exists an \( F_\alpha \) in \( \Theta \) such that \( F_\alpha \subset U \).

Recall that a multifunction (also called multivalued function \([2]\)) \( F \) on a set \( X 
\) into a set \( Y \), denoted by \( F : X \rightarrow Y \), is a relation on \( X \) into \( Y \), i.e. \( F \subset X \times Y \).
Let \( F : X \rightarrow Y \) be a multifunction. The upper and lower inverse of a set \( V \) of \( Y \)
are denoted by \( F^+(V) \) and \( F^-(V) \):

\[ F^+(V) = \{ x \in X \mid F(x) \subset V \} \quad \text{and} \quad F^-(V) = \{ x \in X \mid F(x) \cap V \neq \emptyset \} \]

## 2 Go-compact spaces

We begin with the following notions:

**Definition 1** A point \( x \) in a space \( X \) is said to be a \( g \)-complete accumulation point
of a subset \( S \) of \( X \) if \( Card(S \cap U) = Card(S) \) for each \( U \in GO(X, x) \), where
Card(S) denotes the cardinality of S.

Example 2.1 Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \emptyset, \{a\}, \{a, b, c\}\}$. Observe that both $b$ and $c$ are $g$-complete accumulation points of $\{a, b\}$. Notice that $a$ is not a $g$-complete accumulation point of $\{a, b\}$

Definition 2 In a topological space $X$, a point $x$ is said to be a $g$-adherent point of a filterbase $\Theta$ on $X$ if it lies in the $g$-closure of all sets of $\Theta$.

Observe that the Frechet filter does not satisfy Definition 2. But take a topological space $X$ such that $A \subset X$. Then any point of the $g$-closure of $A$ is a $g$-adherent point of $\Omega = \{U \subset X \mid A \subset U\}$.

Theorem 2.2 A space $X$ is GO-compact if and only if each infinite subset of $X$ has a $g$-complete accumulation point.

Proof. Let the space $X$ be GO-compact and $S$ an infinite subset of $X$. Let $K$ be the set of points $x$ in $X$ which are not $g$-complete accumulation points of $S$. Now it is obvious that for each point $x$ in $K$, we are able to find $U(x) \in \Gamma(X, x)$ such that $Card(S \cap U(x)) \neq Card(S)$. If $K$ is the whole space $X$, then $\Theta = \{U(x) \mid x \in X\}$ is a $\gamma$-cover of $X$. By the hypothesis $X$ is GO-compact, so there exists a finite subcover $\Psi = \{U(x_i)\}$, where $i = 1, 2, \ldots, n$ such that $S \subset \bigcup\{U(x_i) \cap S \mid i = 1, 2, \ldots, n\}$. Then $Card(S) = \max\{Card(U(x_i) \cap S) \mid i = 1, 2, \ldots, n\}$ which does not agree with what we assumed. This implies that $S$ has a $g$-complete accumulation point. Now assume that $X$ is not GO-compact and that every infinite subset $S \subset X$ has a $g$-complete accumulation point in $X$. It follows that there exists a $g$-cover $\Xi$ with no finite subcover. Set $\delta = \min\{Card(\Phi) \mid \Phi \subset \Xi$, where $\Phi$ is a $g$-cover of $X\}$. Fix $\Psi \subset \Xi$ for which $Card(\Psi) = \delta$ and $\bigcup\{U \mid U \in \Psi\} = X$. Let $N$ denote the set of natural numbers. Then by hypothesis $\delta \geq Card(N)$. By well-ordering of $\Psi$ by some minimal well-ordering $^\sim$ we have $Card(\{V \mid V \in \Psi, V \sim U\}) < Card(\{V \mid V \in \Psi\})$. Since $\Psi$ can not have any subcover with cardinality less than $\delta$, then for each $U \in \Psi$ we have $X \neq \bigcup\{V \mid V \in \Psi, V \sim U\}$. For each $U \in \Psi$, choose a point $x(U) \in X \setminus \bigcup\{V \cup \{x(V)\} \mid V \in \Psi, V \sim U\}$. We are always able to do this if not one can choose a cover of smaller cardinality from $\Psi$. If $H = \{x(U) \mid U \in \Psi\}$, then to finish the proof we will show that $H$ has no $g$-complete accumulation point in $X$. Suppose that $z$ is a point of the space $X$. Since $\Psi$ is a $g$-cover of $X$ then $z$ is a point of some set $W$ in $\Psi$. By the fact that $U \sim W$, we have $x(U) \in W$. It follows that $T = \{U \mid U \in \Psi \text{ and } x(U) \in W\} \subset \{V \mid V \in \Psi, V \sim W\}$. But $Card(T) < \delta$. Therefore $Card(H \cap W) < \delta$. But $Card(H) = \delta \geq Card(N)$ since for two distinct points $U$ and $W$ in $\Psi$, we have $x(U) \neq x(W)$. This means that $H$ has no $g$-complete accumulation point in $X$ which contradicts our assumptions. Therefore $X$ is GO-compact.

Theorem 2.3 For a space $X$ the following statements are equivalent:
1. $X$ is GO-compact;
2. Every net in $X$ with a well-ordered directed set as its domain $g$-accumulates to some point of $X$. 
Proof. (1) ⇒ (2): Suppose that $X$ is $GO$-compact and $\xi = \{x_\alpha \mid \alpha \in \Lambda\}$ a net with a well-ordered directed set $\Lambda$ as domain. Assume that $\xi$ has no $g$-adherent point in $X$. Then for each point $x$ in $X$, there exist $V(x) \in GO(X, x)$ and an $\alpha(x) \in \Lambda$ such that $V(x) \cap \{x_\alpha \mid \alpha \geq \alpha(x)\} = \emptyset$. This implies that $\{x_\alpha \mid \alpha \geq \alpha(x)\}$ is a subset of $X \setminus V(x)$. Then the collection $C = \{V(x) \mid x \in X\}$ is a $g$-cover of $X$. By hypothesis of the theorem, $X$ is $GO$-compact and so $C$ has a finite subfamily $\{V(x_i)\}$, where $i = 1, 2, ..., n$ such that $X = \bigcup\{V(x_i)\}$. Suppose that the corresponding elements of $\Lambda$ be $\{\alpha(x_i)\}$, where $i = 1, 2, ..., n$. Since $\Lambda$ is well-ordered and $\{\alpha(x_i)\}$, where $i = 1, 2, ..., n$ is finite, the largest element of $\{\alpha(x_i)\}$ exists. Suppose it is $\{\alpha(x_i)\}$. Then for $\gamma \geq \{\alpha(x_i)\}$, we have $\{x_\delta \mid \delta \geq \gamma\} \subset \bigcap_{i=1}^{n} (X \setminus V(x_i)) = X \setminus \bigcup_{i=1}^{n} V(x_i) = \emptyset$ which is impossible. This shows that $\xi$ has at least one $g$-adherent point in $X$.

(2) ⇒ (1): Now it is enough to prove that each infinite subset has a $g$-complete accumulation point by utilizing Theorem 3.1. Suppose that $S \subset X$ is an infinite subset of $X$. According to Zorn’s Lemma, the infinite set $S$ can be well-ordered. This means that we can assume $S$ to be a net with a domain which is a well-ordered index set. It follows that $S$ has a $g$-adherent point $z$. Therefore $z$ is a $g$-complete accumulation point of $S$. This shows that $X$ is $GO$-compact.

Theorem 2.4 A space $X$ is $GO$-compact if and only if each family of $g$-closed subsets of $X$ with the finite intersection property has a nonempty intersection.

Proof. Straightforward.

Question 2.5 Which condition or conditions should be imposed on a topological space $X$ such that the following statements are equivalent:
(1) $X$ is $GO$-compact;
(2) Each filterbase on $X$ with at most one $g$-adherent point is $g$-convergent.

3 $GO$-compactness and 1-lower and 1-upper $g$-continuous functions

In this section we further investigate properties of $GO$-compactness by 1-lower and 1-upper $g$-continuous functions. We begin with the following notions and in what follows $R$ denotes the set of real numbers.

Definition 3 A function $f : X \rightarrow R$ is said to be 1-lower (resp. 1-upper) $g$-continuous at the point $y$ in $X$ if for each $\lambda > 0$, there exists a $g$-open set $U(y) \in GO(X, y)$ such that $f(x) > f(y) - \lambda$ (resp. $f(x) > f(y) + \lambda$) for every point $x$ in $U(y)$. The function $f$ is 1-lower (resp. 1-upper) $g$-continuous in $X$ if it has these properties for every point $x$ of $X$.

Example 3.1 Take $f : (R, \tau_u) \rightarrow (R, \tau)$, where $\tau_u$ is the usual topology and $\tau$ is the family of sets $\tau = \{(\eta, \infty) \mid \eta \in R\} \cup R$. Such a function is 1-lower $g$-continuous. Now take $f : (R, \tau_u) \rightarrow (R, \sigma)$, where $\tau_u$ is the usual topology and $\sigma$ is the family of sets $\sigma = \{(-\infty, \eta) \mid \eta \in R\} \cup R$. Such a function is 1-upper $g$-continuous.
Theorem 3.2 A function $f : X \to R$ is 1-lower $g$-continuous if and only if for each $\eta \in R$, the set of all $x$ such that $f(x) \leq \eta$ is $g$-closed.

Proof. It is obvious that the family of sets $\tau = \{(\eta, \infty) : \eta \in R\} \cup R$ establishes a topology on $R$. Then the function $f$ is 1-lower $g$-continuous if and only if $f : X \to (R, \tau)$ is $g$-continuous. The interval $(-\infty, \eta]$ is closed in $(R, \tau)$. It follows that $f^{-1}((-\infty, \eta])$ is $g$-closed. Therefore the set of all $x$ such that $f(x) \leq \eta$ is equal to $f^{-1}((-\infty, \eta])$ and thus is $g$-closed.

Corollary 3.3 A subset $S$ of $X$ is GO-compact if and only if the characteristic function $X_S$ is 1-lower $g$-continuous.

Theorem 3.4 A function $f : X \to R$ is 1-upper $g$-continuous if and only if for each $\eta \in R$, the set of all $x$ such that $f(x) \geq \eta$ is $g$-closed.

Corollary 3.5 A subset $S$ of $X$ is GO-compact if and only if the characteristic function $X_S$ is 1-upper $g$-continuous.

Question 3.6 Is it true that if the function $G(x) = \inf_{i \in I} f_i(x)$ exists, where $f_i$, are 1-upper $g$-continuous functions from $X$ into $R$, then $G(x)$ is 1-upper $g$-continuous?

Theorem 3.7 Let $f : X \to R$ be a 1-lower $g$-continuous function, where $X$ is GO-compact. Then $f$ assumes the value $m = \inf_{x \in X} f(x)$.

Proof. Suppose $\eta > m$. Since $f$ is 1-lower $g$-continuous, then the set $K(\eta) = \{x \in X : f(x) \leq \eta\}$ is a non-empty $g$-closed set in $X$ by infimum property. Hence the family $\{K(\eta) : \eta > m\}$ is a collection of non-empty $g$-closed sets with finite intersection property in $X$. By Theorem 2.4 this family has non-empty intersection. Suppose $z \in \bigcap_{\eta > m} K(\eta)$. Therefore $f(z) = m$ as we wished to prove.

4 GO-compactness and $g$-continuous multifunctions

In this section, we give some characterizations of GO-compact spaces by using lower (resp. upper) $g$-continuous multifunctions.

Definition 4 A multifunction $F : X \to Y$ is said to be lower (resp. upper) $g$-continuous if $X \setminus F^{-}(S)$ (resp. $F^{-}(S)$) is $g$-closed in $X$ for each open (resp. closed) set $S$ in $Y$.

For the following two lemmas we shall assume that if $gCl(A) = A$, then $A$ is $g$-closed”.

Lemma 4.1 For a multifunction $F : X \to Y$, the following statements are equivalent:
(1) $F$ is lower $g$-continuous;
(2) If \( x \in F^{-}(U) \) for a point \( x \) in \( X \) and an open set \( U \subset Y \), then \( V \subset F^{-}(U) \) for some \( V \in GO(x) \);

(3) If \( x \notin F^{+}(D) \) for a point \( x \) in \( X \) and a closed set \( D \subset Y \), then \( F^{+}(D) \subset K \) for some \( g \)-closed set \( K \) with \( x \notin K \);

(4) \( F^{-}(U) \in GO(X) \) for each open set \( U \subset Y \).

Proof. (1) \( \Rightarrow \) (4): Let \( U \) be any open set in \( Y \). By (1), \( X - F^{-}(U) \) is \( g \)-closed in \( X \) and hence \( F^{-}(U) \in GO(X) \).

(4) \( \Rightarrow \) (2): Let \( U \) be any open set of \( Y \) and \( x \in F^{-}(U) \). By (4), \( F^{-}(U) \in GO(X) \).

Put \( V = F^{-}(U) \). Then \( V \in GO(X) \) and \( V \subset F^{-}(U) \).

(2) \( \Rightarrow \) (3): Let \( D \) be closed in \( Y \) and \( x \notin F^{+}(D) \). Then \( Y - D \) is open in \( Y \) and \( x \in X - F^{+}(D) = F^{-}(X - D) \).

Therefore, there exists \( V \in GO(x) \) such that \( V \subset F^{-}(U) \). Now, put \( K = X - V \), then \( x \notin K \), \( K \) is \( g \)-closed and \( K = X - V \subset X - F^{-}(Y - D) = F^{+}(D) \).

(3) \( \Rightarrow \) (1): We show that \( F^{+}(H) \) is \( g \)-closed for any closed set \( H \) of \( Y \). Let \( H \) be any closed set and \( x \notin F^{-}(H) \). By (3) there exists a \( g \)-closed set \( K \) such that \( x \notin K \) and \( F^{+}(H) \subset K \); hence \( F^{+}(H) \subset gCl(F^{+}(H)) \subset K \). Since \( x \notin K \), we have \( x \notin gCl(F^{+}(H)) \). This implies that \( gCl(F^{+}(H)) \subset F^{+}(H) \). In general, we have \( F^{+}(H) \subset gCl(F^{+}(H)) \) and hence \( F^{+}(H) = gCl(F^{+}(H)) \). Hence \( F^{+}(H) \) is \( g \)-closed for any closed set \( H \) of \( Y \).

**Lemma 4.2** For a multifunction \( F : X \rightarrow Y \), the following statements are equivalent:

(1) \( F \) is upper \( g \)-continuous;

(2) If \( x \in F^{+}(V) \) for a point \( x \) in \( X \) and an open set \( V \subset Y \), then \( F(U) \subset V \) for some \( U \in GO(x) \);

(3) If \( x \notin F^{-}(D) \) for a point \( x \) in \( X \) and a closed set \( D \subset Y \), then \( F^{-}(D) \subset K \) for some \( g \)-closed set \( K \) with \( x \notin K \);

(4) \( F^{+}(U) \in GO(X) \) for each open set \( U \subset Y \).

Proof. (1) \( \Rightarrow \) (4): Let \( U \) be any open set in \( Y \). Then \( Y - U \) is closed. By (1), \( F^{-}(Y - U) = X - F^{+}(U) \) is \( g \)-closed in \( X \) and hence \( F^{+}(U) \in GO(X) \).

(4) \( \Rightarrow \) (2): Let \( V \) be any open set of \( Y \) and \( x \in F^{+}(V) \). By (4), \( F^{+}(V) \in GO(X) \).

Put \( U = F^{+}(V) \). Then \( U \in GO(X) \) and \( F(U) \subset V \).

(2) \( \Rightarrow \) (3): Let \( D \) be closed in \( Y \) and \( x \notin F^{-}(D) \). Then \( Y - D \) is open and \( x \in X - F^{-}(D) = F^{+}(Y - D) \).

By (2), there exists \( U \in GO(X) \) such that \( F(U) \subset Y - D \). Now, put \( K = X - U \), then \( x \notin K \), \( K \) is \( g \)-closed and \( K = X - U \Rightarrow X - F^{+}(Y - D) = F^{-}(D) \).

(3) \( \Rightarrow \) (1): We show that \( F^{-}(H) \) is \( g \)-closed for any closed set \( H \) of \( Y \). Let \( H \) be any closed set and \( x \notin F^{-}(H) \). By (3), there exists a \( g \)-closed set \( K \) such that \( x \notin K \) and \( F^{-}(H) \subset K \); hence \( F^{-}(H) \subset gCl(F^{-}(H)) \subset K \). Since \( x \notin K \), we have \( x \notin gCl(F^{-}(H)) \). This implies that \( gCl(F^{-}(H)) \subset F^{-}(H) \). In general, we have \( F^{-}(H) \subset gCl(F^{-}(H)) \) and hence \( F^{-}(H) = gCl(F^{-}(H)) \). Hence \( F^{-}(H) \) is \( g \)-closed for any closed set \( H \) of \( Y \).

**Theorem 4.3** The following two statements are equivalent for a space \( X \):

(1) \( X \) is \( GO \)-compact.
(2) Every lower $g$-continuous multifunction from $X$ into the closed sets of a space assumes a minimal value with respect to set inclusion relation.

Proof. (1) $\Rightarrow$ (2): Suppose that $F$ is a lower $g$-continuous multifunction from $X$ into the closed subsets of a space $Y$. We denote the poset of all closed subsets of $Y$ with the set inclusion relation "$\subseteq$" by $\Lambda$. Now we show that $F : X \to \Lambda$ is a lower $g$-continuous function. We will show that $N = F^{-}(\{S \subseteq Y \mid S \in \Lambda \text{ and } S \subseteq C\})$ is $g$-closed in $X$ for each closed set $C$ of $Y$. Let $z \notin N$, then $F(z) \neq S$ for every closed set $S$ of $Y$. It is obvious that $z \in F^{-}(Y \setminus C)$, where $Y \setminus C$ is open in $Y$. By Lemma 4.1 (2), we have $W \subseteq F^{-}(Y \setminus C)$ for some $W \in \text{GO}(z)$. Hence $F(w) \cap (Y \setminus C) \neq \emptyset$ for each $w$ in $W$. So for each $w$ in $W$, $F(w) \setminus C \neq \emptyset$. Consequently, $F(w) \setminus S \neq \emptyset$ for every closed subset $S$ of $Y$ for which $S \subseteq C$. We consider that $W \cap N = \emptyset$. This means that $N$ is $g$-closed. Thus we observe that $F$ assumes a minimal value.

(2) $\Rightarrow$ (1): Suppose that $X$ is not $\text{GO}$-compact. It follows that we have a net $\{x_i \mid i \in \Lambda\}$, where $\Lambda$ is a well-ordered set with no $g$-accumulation point by ([8], Theorem 3.2). We give $\Lambda$ the order topology. Let $M_j = g\text{Cl}\{x_i \mid i \geq j\}$ for every $j$ in $\Lambda$. We establish a multifunction $F : X \to \Lambda$ where $F(x) = \{i \in \Lambda \mid i \geq j_x\}$, $j_x$ is the first element of all those $j$’s for which $x \notin M_j$. Since $\Lambda$ has the order topology, $F(x)$ is closed. By the fact that $\{j_x \mid x \in X\}$ has no greatest element in $\Lambda$, then $F$ does not assume any minimal value with respect to set inclusion relation. We now show that $F^{-}(U) \in \text{GO}(X)$ for every open set $U$ in $\Lambda$. If $U = \Lambda$, then $F^{-}(U) = X$ which is $g$-open. Suppose that $U \subseteq \Lambda$ and $z \in F^{-}(U)$. It follows that $F(z) \cap U \neq \emptyset$. Suppose $j \in F(z) \cap U$. This means that $j \in U$ and $j \in F(z) = \{i \in \Lambda \mid i \geq j_x\}$. Therefore $M_j \geq M_{j_x}$. Since $z \notin M_{j_x}$, then $z \notin M_j$. There exists $W \in \text{GO}(z)$ such that $W \cap \{x_i \mid i \in \Lambda\} = \emptyset$. This means that $W \cap M_j = \emptyset$. Let $w \in W$. Since $W \cap M_j = \emptyset$, it follows that $w \notin M_j$ and since $j_w$ is the first element for which $w \notin M_j$, then $j_w \leq j$. Therefore $j \in \{i \in \Lambda \mid i \geq j_w\} = F(w)$. By the fact that $j \in U$, then $j \in F(w) \cap U$. It follows that $F(w) \cap U \neq \emptyset$ and therefore $w \in F^{-}(U)$. So we have $W \subseteq F^{-}(U)$ and thus $z \in W \subseteq F^{-}(U)$. Therefore $F^{-}(U)$ is $g$-open. This shows that $F$ is lower $g$-continuous which contradicts the hypothesis of the theorem. So the space $X$ is $\text{GO}$-compact.

**Theorem 4.4** The following two statements are equivalent for a space $X$:

1. $X$ is $\text{GO}$-compact.
2. Every upper $g$-continuous multifunction from $X$ into the subsets of a $T_1$-space attains a maximal value with respect to set inclusion relation.

**Proof.** Its proof is similar to that of Theorem 4.3.

The following result concerns the existence of a fixed point for multifunctions on $\text{GO}$-compact spaces.

**Theorem 4.5** Suppose that $F : X \to Y$ is a multifunction from a $\text{GO}$-compact domain $X$ into itself. Let $F(S)$ be $g$-closed for $S$ being a $g$-closed set in $X$. If $F(x) \neq \emptyset$ for every point $x \in X$, then there exists a nonempty, $g$-closed set $C$ of $X$ such that $F(C) = C$. 
Proof. Let $\Lambda = \{ S \subset X \mid S \neq \emptyset, S \in GC(X) \text{ and } F(S) \subset S \}$. It is evident that $x$ belongs to $\Lambda$. Therefore $\Lambda \neq \emptyset$ and also it is partially ordered by set inclusion. Suppose that $\{ S_\gamma \}$ is a chain in $\Lambda$. Then $F(S_\gamma) \subset S_\gamma$ for each $\gamma$. By the fact that the domain is GO-compact and by ([8], Theorem 3.3), $S = \bigcap_\gamma S_\gamma \neq \emptyset$ and also $S \in GC(X)$. Moreover, $F(S) \subset F(S_\gamma) \subset S_\gamma$ for each $\gamma$. It follows that $F(S) \subset S_\gamma$. Hence $S \in \Lambda$ and $S = \inf \{ S_\gamma \}$. It follows from Zorn’s lemma that $\Lambda$ has a minimal element $C$. Therefore $C \in GC(X)$ and $F(C) \subset C$. Since $C$ is the minimal element of $\Lambda$, we have $F(C) = C$.

We close with the following open question:

**Question 4.6** Give a nontrivial example of a GO-compact space?

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