A NOTE ABOUT A THEOREM OF R. HARTE

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Abstract

Let \( A \) and \( B \) be unital Banach algebras and \( T : A \to B \) be a unital continuous homomorphism. Put \( J = \ker T \). Let \( \text{Fred}_T(A) = \{ x \in A | T(x) \text{ is invertible in } B \} \) and \( \text{Fred}_0^T(A) = \{ x + k | x \text{ is invertible in } A, k \in J \} \). In this note, we prove that if \( T \) has Property (F), then \( \text{Fred}_T(A) \cap GL(A) = \text{Fred}_0^T(A) \) iff \( \text{ltsr}(J) = 1 \).

For a normed algebra \( A \) with unit 1, let \( GL(A) \) (resp. \( GL_0(A) \)) denote the group of invertible elements in \( A \) (resp. the connected component of 1 in \( GL(A) \)). If \( A \) is non–unital, we set \( GL(A) = GL(\tilde{A}) \) and \( GL_0(A) = GL_0(\tilde{A}) \), where \( \tilde{A} = \{ \lambda 1 + a | \lambda \in \mathbb{C}, a \in A \} \). For a Banach algebra \( A \), we view \( A^n \) as the set of all \( n \times 1 \) matrices over \( A \).

According to [3], the left topological stable rank of the unital Banach algebra \( A \) is defined as follows:

\[
\text{ltsr}(A) = \min \{ n \in \mathbb{N} | A^n \text{ is dense in } L_{g_m}(A), \forall m \geq n \}
\]

where \( L_{g_n}(A) \) consists of the elements \( (a_1, \cdots, a_n)^T \) in \( A^n \) with \( \sum_{i=1}^{n} b_i a_i = 1 \) for some \( b_1, \cdots, b_n \in A \). If \( A \) is non–unital, we put \( \text{ltsr}(A) = \text{ltsr}(\tilde{A}) \). We have \( \text{ltsr}(A) = 1 \) iff \( GL(A) \) is dense in \( A \) (or \( \tilde{A} \)) (cf. [3]).

Let \( A \) be a unital Banach algebra. Write \( \text{Rg}(A) = \{ a \in A | a \in aAa \} \) and \( \text{Dr}(A) = \{ a \in A | a \in a(GL(A))a \} \) for all regular (generalized invertible) elements and decomposable regular elements of \( A \). Then \( \text{Dr}(A) = \text{Rg}(A) \cap GL(\tilde{A}) \) by [2, Theorem 1.1]). Now let \( B \) be a unital Banach algebra and \( T : A \to B \) be a unital homomorphism (i.e., \( T(1) = 1 \)). Put \( \text{Fred}_T(A) = T^{-1}(GL(B)) \), \( \text{Fred}_0^T(A) = GL(A) + \ker T \). The elements in \( \text{Fred}_T(A) \) are called to be \( T \)-Fredholm and in \( \text{Fred}_0^T(A) \) are called to be \( T \)-Weyl (cf. [1]).

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Let $\mathcal{A}, \mathcal{B}$ be unital Banach algebras and $T$ be a unital continuous homomorphism of $\mathcal{A}$ to $\mathcal{B}$. R. Harte proved in [2] that if $\text{Fred}_T(\mathcal{A}) \subset \text{Rg}(\mathcal{A})$ and $1 + \text{Ker} T \subset \text{Dr}(\mathcal{A})$, then

$$\text{Fred}_T^0(\mathcal{A}) = \text{int}(\text{Fred}_T^0(\mathcal{A})) = \text{Fred}_T(\mathcal{A}) \cap \overline{GL(\mathcal{A})}.$$  \hfill (*)

by means of the equation $\text{Dr}(\mathcal{A}) = \text{Rg}(\mathcal{A}) \cap \overline{GL(\mathcal{A})}$. In this short note, We will show when $\text{Fred}_T^0(\mathcal{A})$ is closed in $\text{Fred}_T(\mathcal{A})$ and prove that if $T$ has Property (F) (see Definition 1 below) the equation (*) holds iff $\text{ltsr} (\text{Ker} T) = 1$.

Throughout the paper, $\mathcal{A}, \mathcal{B}$ are unital Banach algebras and $T: \mathcal{A} \to \mathcal{B}$ is a unital continuous homomorphism.

**Definition 1.** We say $T$ has Property (F) if for every $b \in T(\mathcal{A})$ with $\|1 - b\| < 1$, then $b^{-1} \in T(\mathcal{A})$.

Obviously, if $T(\mathcal{A})$ is closed in $\mathcal{B}$, then $T(\mathcal{A})$ has Property (F). Also, we have

**Proposition 2.** Let $\mathcal{A}, \mathcal{B}$ and $T$ be as above.

1. If $\text{Fred}_T(\mathcal{A}) \subset \text{Rg}(\mathcal{A})$, then $T$ has Property (F);

2. If $T$ has Property (F), then $\text{Fred}_T^0(\mathcal{A})$ is closed in $\text{Fred}_T(\mathcal{A})$.

**Proof.** (1) Let $b \in T(\mathcal{A})$ such that $\|1 - b\| < 1$. Then $b \in GL(\mathcal{B})$. Choose $a \in \mathcal{A}$ such that $b = T(a)$. Since $a \in \text{Fred}_T(\mathcal{A}) \subset \text{Rg}(\mathcal{A})$, there is $a_0 \in \mathcal{A}$ such that $aa_0a = a$ and consequently, $b^{-1} = T(a_0)$. (2) Let $a \in \text{Fred}_T(\mathcal{A})$ and $\{a_n\}_{n=1}^\infty \subset \text{Fred}_T^0(\mathcal{A})$ such that $\lim_{n \to \infty} a_n = a$. Choose $n_0$ such that $\|T(a_{n_0}) - T(a)\| < \frac{1}{2}\|T(a)\|^{-1}$. Then $\|T(a_{n_0})(T(a))^{-1} - 1\| < \frac{1}{2}$. Put $b = T(a_{n_0})(T(a))^{-1} \in GL(\mathcal{B})$. Then $\|b^{-1}\| < \frac{1}{1 - \|1 - b\|} < 2$. Since $b^{-1} \in T(\mathcal{A})$ and $\|b^{-1} - 1\| \leq \|b^{-1}\|\|b - 1\| < 1$, it follows that there is $d \in \mathcal{A}$ such that $b = (b^{-1})^{-1} = T(d)$. Combining this with $b^{-1} \in T(\mathcal{A})$, we can find $c \in \mathcal{A}$ such that $k_1 = ac - 1$ and $k_2 = ca - 1$ are in $\text{Ker} T$. Pick $n_1$ such that $\|a_{n_1} - a\| < \frac{1}{\|c\|}$. Then $\|1 + k_1 - a_{n_1}c\| = \|(a - a_{n_1})c\| < 1$ so that $g = k_1 - a_{n_1}c \in GL(\mathcal{A})$. Therefore

$$a = g^{-1}(k_1 - a_{n_1}c)a = g^{-1}k_1a - g^{-1}a_{n_1}k_2 - g^{-1}a_{n_1} \in \text{Fred}_T^0(\mathcal{A}).$$

$\square$

**Theorem 3.** Let $\mathcal{A}, \mathcal{B}$ be unital Banach algebras and $T: \mathcal{A} \to \mathcal{B}$ be a unital homomorphism with Property (F). Then

$$\text{Fred}_T^0(\mathcal{A}) = \text{int}(\text{Fred}_T^0(\mathcal{A})) = \text{Fred}_T(\mathcal{A}) \cap \overline{GL(\mathcal{A})}$$

iff $\text{ltsr} (\text{Ker} T) = 1$. 
Proof. Since \(GL(A)\) is open in \(A\), \(GL(A) + k\) is open in \(A\) for each \(k \in Ker \ T\). Thus \(Fred^0_T(A) = \{GL(A) + k \mid k \in Ker \ T\}\) is open in \(A\) and hence is open in \(Fred_T(A)\), i.e., \(Fred^0_T(A) = int(Fred^0_T(A))\).

By Proposition 2, when \(T\) has Property (F), \(Fred^0_T(A)\) is closed in \(Fred_T(A)\). Noting that \(GL(A) \subset Fred^0_T(A)\) and \(Fred_T(A) \cap \overline{GL(A)}\) is the closure of \(GL(A)\) in \(Fred_T(A)\), thus \(Fred_T(A) \cap \overline{GL(A)} \subset Fred^0_T(A)\).

We now prove that \(Fred_T(A) \cap \overline{GL(A)} \supset Fred^0_T(A)\) iff \(ltsr(Ker \ T) = 1\).

Suppose that \(ltsr(Ker \ T) = 1\), then for any \(a \in GL(A)\) and \(k \in Ker \ T\),
\[
a + k = a(1 + a^{-1}k) \in a(\overline{GL(Ker \ T)}) \subset \overline{GL(A)},
\]
i.e., \(Fred^0_T(A) \subset Fred_T(A) \cap \overline{GL(A)}\).

Conversely, for any \(k \in Ker \ T\) and any \(\epsilon \in (0,1)\), there is \(x_\epsilon \in GL(A)\) such that \(\|1 + k - x_\epsilon\| < \frac{\epsilon}{4(1 + \|1 + k\|)}\left(\frac{1}{2}\right)\). Put \(a_\epsilon = x_\epsilon - k\). Then \(a_\epsilon \in GL(A)\) and \(\|a_\epsilon^{-1}\| < \frac{1}{1 - \|1 - a_\epsilon\|} < 2\). Set \(z_\epsilon = a_\epsilon^{-1}x_\epsilon\). Then \(z_\epsilon \in GL(A)\), \(T(z_\epsilon) = T(z_\epsilon^{-1}) = 1\), i.e., \(z_\epsilon \in GL(Ker \ T)\) and furthermore,
\[
\|1 + k - z_\epsilon\| \leq \|1 + k - x_\epsilon\| + \|a_\epsilon^{-1}\||1 - a_\epsilon||x_\epsilon|| < \epsilon.
\]

Now let \(x = \lambda 1 + z \in \overline{Ker \ T}\). If \(\lambda = 0\), we put \(x_\epsilon = \epsilon 1 + z = \epsilon(1 + \epsilon^{-1}z)\). Then \(\|x - x_\epsilon\| < \epsilon\) and \(x_\epsilon \in \overline{GL(Ker \ T)}\). So \(x \in \overline{GL(Ker \ T)}\). If \(\lambda \neq 0\), then \(x = \lambda(1 + \lambda^{-1}z) \in \overline{GL(Ker \ T)}\). Therefore, \(ltsr(Ker \ T) = 1\).

We conclude the paper with following two examples:

Example 4. Let \(X\) be a Banach space and let \(B(X)\) (resp. \(K(X)\)) denote the Banach algebra of all bounded linear operators (resp. compact operators) on \(X\). Let \(T\) be the canonical homomorphism of \(B(X)\) onto \(B(X)/K(X)\). Then \(Ker \ T = K(X)\). Using the fact that every nonzero point in the spectrum of a compact operator is isolated, we can deduce that \(ltsr(K(X)) = 1\). So by Theorem 3, \(Fred^0_T(B(X)) = int(Fred^0_T(B(X))) = Fred_T(B(X)) \cap \overline{GL(B(X))}\).

Example 5. Let \(A = C(\mathbb{D})\) and \(B = (S^1)\). Let \(T\) be the homomorphism from \(A\) onto \(B\) given by restriction \(T(f)(z) = f(z), \forall z \in S^1, f \in C(\mathbb{D})\). Since \(Ker \ T \cong C_0(\mathbb{R}^2)\) and \(Ker \ T \cong C(S^1)\), it follows from [3, Proposition 1.7] that \(ltsr(Ker \ T) = 2\). By Theorem 3, \(Fred^0_T(A)\) is both open and closed in \(Fred_T(A)\) and \(Fred_T(A) \cap GL(A) \not\subset Fred^0_T(A)\).

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References


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