ON CONVERGENCE AND DIVERGENCE OF FOURIER EXPANSIONS ASSOCIATED TO JACOBI MEASURE WITH MASS POINTS

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Abstract

We prove the failure of a.e. convergence of the Fourier expansion in terms of the orthonormal polynomials with respect to the measure $(1-x)\alpha(1+x)\beta \, dx + M\delta_{-1} + N\delta_1$, where $\delta_t$ is the delta function at a point $t$ and $M > 0, N > 0$. Lebesgue norms of Koornwinder's Jacobi-type polynomials are applied to obtain a new proof of necessary conditions for mean convergence.

1 Introduction

Let $\omega_{\alpha,\beta}(x) = (1-x)\alpha(1+x)\beta$, $(\alpha, \beta > -1)$, be the Jacobi weight on the interval $[-1, 1]$. In [6] T. H. Koornwinder introduced the polynomials $\{P_n^{(\alpha,\beta,M,N)}(x)\}_{n=0}^\infty$ which are orthogonal on the interval $[-1, 1]$ with respect to the measure

$$d\mu(x) = \frac{\Gamma(\alpha + \beta + 2)}{2^{\alpha + \beta + 1}\Gamma(\alpha + 1)\Gamma(\beta + 1)}\omega_{\alpha,\beta}(x)dx + M\delta_{-1} + N\delta_1,$$

where $\alpha > -1, \beta > -1$, and $M, N \geq 0$. They are called Koornwinder’s Jacobi-type polynomials. We denote the orthonormal Koornwinder’s Jacobi-type polynomial by $p_n^{(\alpha,\beta,M,N)}$, which differs from $P_n^{(\alpha,\beta,M,N)}$ by normalization constant (see [14, p. 81]). For $M = N = 0$, denoted by $p_n^{(\alpha,\beta)}$, we have the classical Jacobi orthonormal polynomials (see [13, Chapter IV]). It is known that, unlike the Jacobi orthonormal polynomials, the polynomials $p_n^{(\alpha,\beta,M,N)}$ for $M > 0, N > 0$ decay at the rate of $n^{-\alpha-3/2}$ and $n^{-\beta-3/2}$ at the end points 1 and $-1$.

We shall say that $f(x) \in L^p(d\mu)$ if $f(x)$ is measurable on the $[-1, 1]$ and $\|f\|_{L^p(d\mu)} < \infty$, where

$$\|f\|_{L^p(d\mu)} = \begin{cases} \left(\int_{-1}^1 |f(x)|^p d\mu(x)\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \operatorname{esssup}_{-1 < x < 1} |f(x)| & \text{if } p = \infty. \end{cases}$$

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For $f \in L^1(d\mu)$, the Fourier expansions in Koornwinder’s Jacobi-type polynomials is
\[ \sum_{k=0}^{\infty} \hat{f}(k)p_k^{(\alpha,\beta,M,N)}(x) \] (1.1)
where the Fourier coefficients are
\[ \hat{f}(k) = \int_{-1}^{1} f(x)p_k^{(\alpha,\beta,M,N)}(x)d\mu(x) \]
\[ = \frac{\Gamma(\alpha + \beta + 2)}{2^{\alpha+\beta+1}\Gamma(\alpha + 1)\Gamma(\beta + 1)} \int_{-1}^{1} f(x)p_k^{(\alpha,\beta,M,N)}(x)\omega_{\alpha,\beta}(x)dx \]
\[ + Mf(-1)p_k^{(\alpha,\beta,M)}(-1) + Nf(1)p_k^{(\alpha,\beta,M,N)}(1). \] (1.2)

The Cesàro means of order $\rho$ of the expansion (1.1) are defined by (see [15, p. 76-77], [9])
\[ \sigma_n^\rho f(x) = \sum_{k=0}^{n} \frac{A_n}{A_n^\rho} \hat{f}(k)p_k^{(\alpha,\beta,M,N)}(x), \]
where $A_n^\rho = \binom{k+\rho}{k}$.

In 1972 Pollard [11] raised the following question: Is there an $f \in L^{4/3}(dx)$ whose Fourier-Legendre expansion diverges almost everywhere? This problem was solved by Meaney [8]. Furthermore, he proved that this is a special case of divergence result for series of Jacobi polynomials.

This paper is a continuation of [1]. We will prove that, for $\alpha > -1/2$ and $p_0 = (4\alpha + 4)/(2\alpha + 3)$, there are functions $f \in L^{p_0}(d\mu)$ whose Fourier expansions in terms of the $\{p_n^{(\alpha,\beta,M,N)}\}_{n=0}^{\infty}$ are divergent almost everywhere on $[-1,1]$. Moreover we show that, for $1 < p < p_0$ and $0 < \rho < 2/p - 3/2$, there are functions $f \in L^p(d\mu)$ with almost everywhere divergent Cesàro means of order $\rho$. We also find the necessary conditions for the convergence in $L^p(d\mu)$ norm of Fourier expansion (1.1).

In order to obtain it, previously, we need some estimates for Koornwinder’s Jacobi-type orthonormal polynomials. The representation of the $p_n^{(\alpha,\beta,M,N)}$ in terms of $p_n^{(\alpha,\beta)}$, a strong asymptotic on $(-1,1)$, a Mehler-Heine type formula, Lebesgue norms of $p_n^{(\alpha,\beta,M,N)}$ are derived.

## 2 Estimates for Koornwinder’s Jacobi-type polynomials

The goal of this section is to obtain estimates and asymptotic properties on $[-1,1]$ for the orthonormal polynomials $p_n^{(\alpha,\beta,M,N)}$. Throughout this paper positive constants are denoted by $c$, $c_1$, … and they may vary at every occurrence. The notation $u_n \asymp v_n$ means that the sequence $u_n/v_n$ converges to 1 and notation $u_n \sim v_n$ means $c_1u_n \leq v_n \leq c_2u_n$ for sufficiently large $n$. 
Proposition 2.1. The representation of the \( p_{n}^{(\alpha,\beta,M,N)} \) in terms of \( p_{n}^{(\alpha,\beta,M,0)} \) is
\[
p_{n}^{(\alpha,\beta,M,N)}(x) = A_{n}p_{n}^{(\alpha,\beta,M,0)}(x) + B_{n}(x-1)p_{n-1}^{(\alpha,\beta,2M,0)}(x),
\] (2.1)
where
\[
A_{n} \sim cn^{-2\alpha-2}, \quad B_{n} \sim 1.
\] (2.2)

Proof. Let \( \{P_{n}^{1}\}_{n=0}^{\infty} \) be the orthonormal polynomials with respect to the measure (see proof of the Proposition 6 in [4])
\[
(x-1)^{2}[\omega_{\alpha,\beta}(x)dx + M\delta_{-1}] = \omega_{\alpha+2,\beta}(x)dx + 4M\delta_{-1}.
\]
Therefore \( P_{n}^{1} = p_{n}^{(\alpha+2,\beta,4M,0)} \). From [4, Proposition 4] it follows
\[
p_{n}^{(\alpha,\beta,M,N)}(x) = A_{n}p_{n}^{(\alpha,\beta,M,0)}(x) + B_{n}(x-1)p_{n-1}^{(\alpha,\beta,4M,0)}(x),
\]
where
\[
\lim_{n \to \infty} A_{n}L_{n-1}(1,1) = \frac{1}{\lambda(1)+N}
\]
\[
\lim_{n \to \infty} B_{n} = \frac{N}{\lambda(1)+N}
\]
\[
\lambda(1) = \lim_{n \to \infty} \frac{1}{L_{n}(1,1)}.
\]
Since (see [1, (3)] and [13, (4.5.8)])
\[
L_{n}(1,1) = \sum_{i=0}^{n} p_{i}^{(\alpha,\beta,M,0)}(1)p_{i}^{(\alpha,\beta,M,0)}(1) \sim cn^{2\alpha+2}
\]
we get (2.2).

Combining the above proposition with [1, (7)] we obtain:

Corollary 2.1. The representation of the \( p_{n}^{(\alpha,\beta,M,N)} \) in terms of \( p_{n}^{(\alpha,\beta)} \) is
\[
p_{n}^{(\alpha,\beta,M,N)}(x) = a_{n}p_{n}^{(\alpha,\beta)}(x) + b_{n}(x+1)p_{n-1}^{(\alpha,\beta+2)}(x)
\]
\[
+ c_{n}(x-1)p_{n-1}^{(\alpha,\beta+2)}(x) + d_{n}(x^{2}-1)p_{n-2}^{(\alpha,\beta+2,2)}(x)
\]
where
\[
a_{n} \sim cn^{-2\alpha-2\beta-4}, \quad b_{n} \sim cn^{-2\alpha-2}, \quad c_{n} \sim cn^{-2\beta-2}, \quad d_{n} \sim 1.
\]

The following proposition establishes a strong asymptotic on \((-1,1)\) for \( p_{n}^{(\alpha,\beta,M,N)} \).

Proposition 2.2. For \( \theta \in [\epsilon, \pi - \epsilon] \) and \( \epsilon > 0 \)
\[
p_{n}^{(\alpha,\beta,M,N)}(x) \sim \frac{p_{n}^{(\alpha,\beta,M,N)}(1-x) - \alpha/2 - 1/4}{(1+x)^{-\beta/2 - 1/4}}
\]
\[
\times \cos(k\theta + \gamma) + O(n^{-1}),
\]
where \( x = \cos \theta, k = n + (\alpha + \beta + 1)/2, \gamma = -(\alpha + 1)/2, \pi \) and \( \lim_{n \to \infty} p_{n}^{(\alpha,\beta,M,N)} = \sqrt{2/\pi} \).
Proof. From (2.1) and [1, Lemma 1]

\[ p_n^{(\alpha, \beta, M, N)}(x) = [A_n s_n^{\alpha, \beta} + B_n s_n^{\alpha+2, \beta}](1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4} \]

\[ \times \cos(k\theta + \gamma) + [A_n + B_n(x-1)]O(n^{-1}), \]

\[ \lim_{n \to \infty} s_n^{\alpha, \beta} = \sqrt{2/\pi}. \]

Now taking into account (2.2), the result follows.

Next we give a Mehler-Heine type formula of the polynomials \( p_n^{(\alpha, \beta, M, N)} \).

Proposition 2.3. Uniformly on compact subsets of \( \mathbb{C} \)

\[ \lim_{n \to \infty} n^{-\alpha-1/2} p_n^{(\alpha, \beta, M, N)} \left( \cos \frac{z}{n} \right) = -2^{\frac{\alpha+\beta}{2}} z^{-\alpha} J_{\alpha+2}(z), \]

where \( J_\alpha \) is the Bessel function of order \( \alpha \).

Proof. The Mehler-Heine type formula for Jacobi orthonormal polynomials \( p_n^{(\alpha, \beta)} \left( \cos \frac{z}{n+j} \right) \), \( j \in \mathbb{N} \cup 0 \), is (see [13, Theorem 8.1.1])

\[ \lim_{n \to \infty} n^{-\alpha-1/2} p_n^{(\alpha, \beta)} \left( \cos \frac{z}{n+j} \right) = 2^{\frac{\alpha+\beta}{2}} (z/2)^{-\alpha} J_\alpha(z), \]

uniformly on compact subsets of \( \mathbb{C} \). Although the above formula in [13, Theorem 8.1.1] is for \( j = 0 \), it can be shown that this formula is also true for any fixed \( j \in \mathbb{N} \).

By Corollary 2.1 we have

\[ n^{-\alpha-1/2} p_n^{(\alpha, \beta, M, N)} \left( \cos \frac{z}{n} \right) = a_n n^{-\alpha-1/2} p_n^{(\alpha, \beta)} \left( \cos \frac{z}{n} \right) \]

\[ + b_n \left( \cos \frac{z}{n} + 1 \right) n^{-\alpha-1/2} p_{n-1}^{(\alpha, \beta+2)} \left( \cos \frac{z}{n} \right) \]

\[ - 2c_n \sin^2 \frac{z}{2n} n^{-\alpha-1/2} p_{n-1}^{(\alpha+2, \beta)} \left( \cos \frac{z}{n} \right) \]

\[ - d_n \sin^2 \frac{z}{n} n^{-\alpha-1/2} p_{n-2}^{(\alpha+2, \beta+2)} \left( \cos \frac{z}{n} \right). \]

Now, using the estimates for the coefficients \( a_n, b_n, c_n \) and \( d_n \), the result follows.

The proofs of main results are based on following proposition.

Proposition 2.4. Let \( \alpha \geq -1/2 \) and \( M, N > 0 \). For \( 1 \leq q < \infty \)

\[ \int_0^1 (1-x)^\alpha |p_n^{(\alpha, \beta, M, N)}(x)|^q dx \sim \begin{cases} \frac{c}{q} & \text{if } 2\alpha > q\alpha - 2 + q/2, \\ \log n & \text{if } 2\alpha = q\alpha - 2 + q/2, \\ n^{q\alpha+q/2-2\alpha-2} & \text{if } 2\alpha < q\alpha - 2 + q/2. \end{cases} \]

Proof. The upper estimates has been proved in [1, Theorem 1]. In order to prove the lower estimate, we follow the same line as in [13, Theorem 7.34] (see also [1, Theorem 2]), by using the Proposition 2.3 and [12, Lemma 2.1].
By using this proposition, [6, (2.5)] and [1, (3),(4)], we obtain:

**Corollary 2.2.** Let \( \alpha \geq \beta \geq -1/2 \) and \( \alpha > -1/2 \). For \( q_0 = \frac{4\alpha+4}{2\alpha+1} \),

\[
\|p_n^{(\alpha,\beta,M,N)}(x)\|_{L^q(d\mu)} \sim \begin{cases} 
  c & \text{if } 1 \leq q < q_0, \\
  (\log n)^{\frac{1}{q_0}} & \text{if } q = q_0, \\
  n^{\alpha+1/2-2(\alpha+1)/q} & \text{if } q_0 < q < \infty.
\end{cases}
\]

### 3 Divergence almost everywhere

Suppose that the expansion (1.1) converges on a subset \( E \) of positive measure in \([-1,1]\). Then

\[
c_n(f)p_n^{(\alpha,\beta,M,N)}(x) \to 0, \quad x \in E. \tag{3.1}
\]

From Egorov's theorem it follows that there is a subset \( E_1 \subset E \) of positive measure \( E \) such that (3.1) holds uniformly for \( x \in E_1 \). Therefore, from Proposition 2.2, we have

\[
n^{-\delta}c_n(f)\left(\cos(k\theta + \gamma) + O(n^{-1})\right) \to 0
\]

uniformly for \( x = \cos \theta \in E_1 \). By a variant of the Cantor-Lebesgue Theorem, cf. [9, Subsection 1.5], this implies

\[
c_n(f) \to 0. \tag{3.2}
\]

Now we are in position to prove our first main result

**Theorem 3.1.** Let \( \alpha > -1/2 \) and \( \beta > -1 \). There is a function \( f \) in \( L^{p_0}(d\mu) \), supported in \([0,1]\), such that its Fourier expansion (1.1) diverges for almost every \( x \in [-1,1] \).

**Proof.** For every function \( f \in L^1(d\mu) \) the Fourier coefficients (1.2) can be written as

\[
c_n(f) = c'_n(f) + Mf(-1)p_n^{(\alpha,\beta,M,N)}(-1) + Nf(1)p_n^{(\alpha,\beta,M,N)}(1), \tag{3.3}
\]

where

\[
c'_n(f) = \frac{\Gamma(\alpha + \beta + 2)}{2^{\alpha+\beta+1} \Gamma(\alpha + 1) \Gamma(\beta + 1)} \int_{-1}^{1} f(x)p_n^{(\alpha,\beta,M,N)}(x)\omega_{\alpha,\beta}(x)dx.
\]

The uniform boundedness principle and Proposition 2.4 yields the existence of functions \( f \in L^{p_0}(d\mu) \), supported on \([0,1]\), such that the linear functional \( c'_n(f) \) satisfies

\[
\frac{c'_n(f)}{(\log n)^{\frac{1}{q_0}}} \to \infty, \quad \text{as } n \to \infty.
\]

Hence, from (3.3) and [1, (3),(4)], we obtain

\[
\frac{c_n(f)}{(\log n)^{\frac{1}{q_0}}} \to \infty, \quad \text{as } n \to \infty,
\]

which contradict (3.2). \( \square \)
Now we show that, for some values of $\delta$, there are functions with a.e. divergent Cesàro means.

**Theorem 3.2.** Let given numbers $\alpha$, $\beta$, $p$, and $\delta$ be such that $\alpha > -1/2$; $\beta > -1$;

\[
1 < p < \frac{4(\alpha + 1)}{2\alpha + 3},
\]

\[0 \leq \delta < \frac{2\alpha + 2}{p} - \frac{2\alpha + 3}{2}.
\]

There is an $f \in L^p(d\mu)$, supported in $[0,1]$, whose Cesàro means $\sigma^f_N f(x)$ is divergent almost everywhere on $[-1,1]$.

**Proof.** From Egorov’s theorem and [9, Lemma 1.1] (see also [15, Theorem 3.1.22]) it follows that if the series (1.1) is Cesàro summable of order $\delta$ on a set $E$ of positive measure in $[-1,1]$ then there is a subset $E_1 \subset E$ of positive measure where

\[|n^{-\delta} c_n(f)p_n^{(\alpha,\beta,M,N)}(x)| \leq c\]

uniformly for $x \in E_1$. Hence, from Proposition 2.2, we have

\[|n^{-\delta} c_n(f)(\cos(k\theta + \gamma) + O(n^{-1})))| \leq c\]

uniformly for $\cos \theta \in E_1$. Using again the Cantor-Lebesgue Theorem we obtain

\[
\left|\frac{c_n(f)}{n^\delta}\right| \leq c, \quad \forall n \geq 1. \tag{3.4}
\]

Suppose that

\[
\delta < \frac{2\alpha + 2}{p} - \frac{2\alpha + 3}{2}.
\]

For $q$ conjugate of $p$\n
\[
\delta < \alpha + \frac{1}{2} - \frac{2\alpha + 2}{q}.
\]

From the argument given in the [9, Subsection 1.4] and Proposition 2.4, for the linear functional $c_n'(f)$, it follows that there is an $f \in L^p(d\mu)$, supported on $[0,1]$, such that

\[
\frac{c_n'(f)}{n^\delta} \to \infty, \quad \text{as } n \to \infty.
\]

So, from (3.3) and [1, (3),(4)], it follows that

\[
\frac{c_n(f)}{n^\delta} \to \infty, \quad \text{as } n \to \infty.
\]

Combining the above results with (3.4) it follows that, for this $f$, the $\sigma^f_N f(x)$ diverges almost everywhere.

**Remark 3.1.** Using formulae in [2], which relates the Riesz and Cesàro means of order $\delta \geq 0$, we conclude that the Theorem 3.2 also holds for the Riesz means.
4 Necessary conditions for the norm convergence

Let $S_n f$ be the $n$-th partial sum of the expansion (1.1)

$$S_n f(x) = \sum_{k=0}^{n} \hat{f}(k)p_k^{(\alpha, \beta, M, N)}(x)$$

If $\alpha \geq \beta \geq -1/2$ and $\alpha > -1/2$, then (see [3], [5], and [7] in a more general framework)

$$\|S_n f\|_{L^p(d\mu)} \leq C\|f\|_{L^p(d\mu)} \quad \forall n \geq 0, \forall f \in L^p(d\mu)$$

if and only if $p$ belongs to the open interval $(p_0, q_0)$.

Now we will give a new proof of the following theorem.

**Theorem 4.1.** Let $\alpha \geq \beta \geq -1/2$ and $\alpha > -1/2$. If there exists a constant $c > 0$

$$\|S_n f\|_{L^p(d\mu)} \leq c\|f\|_{L^p(d\mu)}$$

for every $f \in S_p$ and $n \geq 0$, then $p \in (p_0, q_0)$

**Proof.** For the proof, we apply the same argument as in [10] (see also [12]). Assume that (4.1) holds true. Then

$$\|\hat{f}(n)p_n^{(\alpha, \beta, M, N)}(x)\|_{L^p(d\mu)} \leq 2c\|f\|_{L^p(d\mu)}.$$

Therefore

$$\|p_n^{(\alpha, \beta, M, N)}(x)\|_{L^p(d\mu)} \|p_n^{(\alpha, \beta, M, N)}(x)\|_{L^q(d\mu)} < \infty,$$

where $p$ is the conjugate of $q$. By Corollary 2.2, it follows that the last inequality holds if and only if $p \in (p_0, q_0)$.

The proof of Theorem 4.1 is complete. \qed

**Remark 4.1.** Using the symmetry properties [6, (2.5)], we get the same results as above with $\alpha$ replaced by $\beta$.

**References**

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